SOME REMARKS ON CONNECTED SETS

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This note will consist of a few disconnected remarks on connected sets.

i. Swingle[1] raised the question whether the plane is the sum of c disjoint biconnected sets. The answer as we shall show is affirmative.

First we construct a biconnected set A with a dispersion point x such that any two points of $A-x$ can be separated. (The first such set was constructed by Wilder.[2] Our construction will be very similar to that of Burton Jones.[3])

Our biconnected set will contain the origin $\mathfrak{O}$, at most one other point on lines through the origin with irrational slopes, and no point other than the origin on lines with rational slopes. Further it will contain at least one point on every cut of the plane. It is easy to see that such a set exists. Every cut of the plane contains a closed subset which also cuts the plane, and the power of closed sets is known to be c. Let us well order the closed cuts $C_1, C_2, \cdots, C_\gamma, \cdots, \gamma<\Omega_\alpha$, where Ω_α is the least ordinal number of power c. We construct A as follows: $\mathfrak{O}$ belongs to A. We shall choose a point x_γ on C_γ and we shall have $A=\bigcup x_\gamma$. We shall determine x_γ by transfinite induction. Suppose we have already determined x_δ, $\delta<\gamma$, we determine x_γ as follows: if C_γ contains $\mathfrak{O}$ then $x_\gamma=\mathfrak{O}$. If C_γ does not contain $\mathfrak{O}$ then clearly C_γ has to intersect c lines through $\mathfrak{O}$. Therefore we can find a point $x_\gamma\in C_\gamma$ such that $(\mathfrak{O}, x_\gamma)$ has irrational slope and does not go through any other point x_δ, $\delta<\gamma$. (We denote by (a, b) the line through the points a and b.) This way we construct A. Clearly A is biconnected. First of all A is connected since it intersects every cut of the plane. Also any two points of $A-\mathfrak{O}$ can be separated since the two points x_1 and x_2, say, are on different irrational lines through the origin, and a rational line, which of course does not intersect $A-\mathfrak{O}$, will separate them. This completes the proof. The origin we call the center of A.

Now we shall split the plane into the sum of c such disjoint sets A_γ, $\gamma<\Omega_\alpha$, Ω_α the smallest ordinal of power c. Let us well order the points x_γ of the plane and the closed cuts C_γ of the plane. Our first step is to select x_1 as the center of A_1 and a suitable point of A_1 on C_1.

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[1] P. M. Swingle, Amer. J. Math. vol. 54 (1932) p. 532.

[2] R. L. Wilder, Bull. Amer. Math. Soc. vol. 33 (1927) p. 423.

[3] Burton Jones, Bull. Amer. Math. Soc. vol. 48 (1942) p. 115.