LIMIT POINTS OF SUBSEQUENCES

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1. **Introduction.** In a previous paper [2],[2] it was shown that for simple sequences of real numbers, divergence of a sequence implies divergence of almost every subsequence. The proof given there required in an essential way that the space be metric. The purpose of this note is to show that the above result holds for multiple sequences in an L^* space. If the space is compact separable metric, even more is true: *the set of limit points of almost every subsequence coincides with the set of limit points of the original sequence.*

2. **Notation.** We denote by $x=x[n]=x[n_1, n_2, \cdots, n_r]$, where $n_k=1, 2, \cdots$, an arbitrary r-tuple sequence with terms in a space M. Likewise, $x'=x_\lambda[n]=x[\lambda_1(n_1), \lambda_2(n_2), \cdots, \lambda_r(n_r)]$ will denote an arbitary subsequence of x; here, for any fixed k, $\lambda_k(1), \lambda_k(2), \cdots$ form an increasing sequence of integers. This is the natural generalization of subsequences $x_{\lambda n}$ of the simple sequence x_n.

Let $\mathcal{S}$ be the set of all sequences $s=(s_\alpha)$ composed of 0's and 1's, containing infinitely many 1's. This represents in the usual manner the class of subsequences of a simple sequence [1, p. 788] that is $s_n=1$ if x_n is chosen, and 0 if x_n is omitted. A product measure can be defined in $\mathcal{S}$ [4, p. 420; 5, p. 144].[3]
Then

$$\mathfrak{S} = \mathcal{S} \times \mathcal{S} \times \cdots \times \mathcal{S} \qquad (r \text{ factors})$$

is the class of all subsequences of an r-tuple sequence. Measure is defined in $\mathfrak{S}$ as the product measure over $\mathcal{S}$.

We assume that some definition of convergence is given for sequences in M. Limit points are then defined as

$$Px = \hat{p}[\lim x' = p \text{ for some subsequence } x' \text{ of } x].$$

We recall the defining conditions of the Fréchet limit spaces [3, p. 77]:

(L) $\lim x=p$ *implies* $\lim x'=p$ *for every subsequence* x' *of* x.

(L^*) $\lim x=p$ *if and only if* $p\in Px'$ *for every subsequence* x' *of* x.

L^* is the "star-convergence" of Urysohn.

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[2] Numbers in brackets refer to the references listed at the end of the paper.

[3] $\mathcal{S}\subseteq\mathcal{S}^*=S\times S\times S\times S\times \cdots$, where $S=\{0, 1\}$. Both $\mathcal{S}$ and $\mathcal{S}^*$ have measure 1. $\mathcal{S}$ can be mapped, with measure preserved, on the real numbers $0<t\leqq 1$.