POWERS OF HOMEOMORPHISMS WITH ALMOST PERIODIC PROPERTIES

W. H. GOTTSCHALK[1]

Let X be a topological space (an "accessible space," a "1-space," or a "T_1-space" in the terminology of Fréchet, Kuratowski, or Alexandroff-Hopf, respectively) and let $f(X)=X$ be a homeomorphism. We use the following terminology, which was suggested by G. A. Hedlund and which is to be carefully distinguished from those terminologies used by Birkhoff, Ayres, Whyburn, and others. A point x of X is said to be *recurrent* under f provided that to each neighborhood U of x there corresponds a positive integer n such that $f^n(x) \in U$. The mapping f is said to be *pointwise recurrent* provided that each point of X is recurrent under f. A point x of X is said to be *almost periodic* under f provided that to each neighborhood U of x there corresponds a monotone increasing sequence $n_1, n_2, \cdots$ of positive integers with the properties that the numbers $n_{i+1}-n_i$ $(i=1, 2, \cdots)$ are uniformly bounded and $f^{n_i}(x) \in U$ $(i=1, 2, \cdots)$. The mapping f is said to be *pointwise almost periodic* provided each point of X is almost periodic under f. Following Birkhoff [1, p. 198],[2] a subset Y of X is said to be *minimal* under f provided that Y is nonvacuous, closed and invariant under f, that is, $f(Y)=Y$, and furthermore Y does not contain a proper subset with these properties. For $x \in X$, the set $\sum_{n=-\infty}^{+\infty} f^n(x)$ is called the *orbit* of x under f and the set $\sum_{n=0}^{+\infty} f^n(x)$ is called the *semi-orbit* of x under f. A *decomposition* of X is a collection of nonvacuous pairwise disjoint closed subsets of X which fill up X.

THEOREM 1. *If $x \in X$ is recurrent under f, then x is also recurrent under f^n for every positive integer n.*

PROOF. We make use of an induction. The theorem is true for $n=1$. Let m be any positive integer. Assume the theorem is true for $n \leqq m$. We now show the theorem is true for $n=m+1=k$.

We may suppose without loss of generality that X is the closure of the semi-orbit of x under f, for this set is invariant under f. Define X_i $(i=0, 1, \cdots, k)$ to be the closure of the semi-orbit of $f^i(x)$ under f^k. It is readily verified that $f(X_i)=X_{i+1}$ $(i=0, 1, \cdots, k-1)$,

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[2] Numbers in brackets refer to the bibliography at the end of the paper.