MODULARITY IN BIRKHOFF LATTICES

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The purpose of this note is to identify upper semi-modular lattices originally defined by G. Birkhoff[1] and subsequently studied by Dilworth[2] with those M-symmetric lattices[3] (introduced independently by the author without assumption of chain conditions) which satisfy a condition of finite dimensionality.

The definitions and notations are these. In a lattice L, $a \succ b (b \prec a)$ means that a "covers" b, that is, $a > b$, together with $a \geqq x \geqq b$ implies $x = a$ or $x = b$; $(b, c)M$ means $(a+b)c = a+bc$ for every $a \leqq c$ (where $a+b$, ab are the "join" and "meet" respectively of a, b). We say that L is *M-symmetric* if the binary relation M is symmetric; L is a *Birkhoff lattice* if

$$(1) \qquad a, b \succ ab \text{ implies } a + b \succ a, b;$$

L is *of finite-dimensional type*[4] if for every $a < b$ there exists a finite "principal chain"

$$a_1 \prec a_2 \prec \cdots \prec a_n,$$

with $a_1 = a$, $a_n = b$. When a, b satisfy this condition for a specific n, we say that *b is $n-1$ steps over a*.

The properties of the relation M are given in part in a previous paper.[5] Additional properties needed here are contained in the following lemma.

LEMMA 1. *Suppose $b, c \in L$. Then*
(a) *$(b, c)M$ if and only if $bc \leqq a \leqq c$ implies $(a+b)c = a$;*
(b) *if $(b, c)M$, then $(b', c')M$ for $bc \leqq b' \leqq b$, $bc \leqq c' \leqq c$.*

PROOF. The forward implication in (a) is obvious. To prove the

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[1] G. Birkhoff, *Lattice theory*, Amer. Math. Soc. Colloquium Publications vol. 25, New York, 1940, p. 62.

[2] R. P. Dilworth, *Ideals in Birkhoff lattices*, Trans. Amer. Math. Soc. vol. 49 pp. 325–353; also *The arithmetical theory of Birkhoff lattices*, Duke Math. J. vol. 8 (1941) pp. 286–299.

[3] L. R. Wilcox, *Modularity in the theory of lattices*, Ann. of Math. vol. 40 (1939) pp. 490–505; see also *A note on complementation in lattices*, Bull. Amer. Math. Soc. vol. 48 (1942) pp. 453–457.

[4] This property is weaker than finite dimensionality as used by Birkhoff (loc. cit. p. 11), even if 0 and 1 exist.

[5] L. R. Wilcox, *Modularity in the theory of lattices*, pp. 491–495.