TOPICS IN THE THEORY OF ABELIAN GROUPS. I
DIVISIBILITY OF HOMOMORPHISMS

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1. Introduction. The theory of character groups of Abelian groups has in recent years become of great importance, especially through applications to topology and algebra. The character group of G is the group H of homomorphisms of G into the real numbers mod 1. Extending this, we may consider the group $H = \text{Hom}(G, Z)$ of homomorphisms of G into a third group Z, or more generally, a "pairing" of groups H and G into Z: a multiplication $h \cdot g = z$, satisfying both distributive laws.

Of course the duality theorems for character groups will not hold in the more general cases; but, under certain conditions, substitutes may hold. We expect in later notes to give various facts about pairings, and the closely associated problem of divisibility by integers. In the present note, we answer the question of when a homomorphism of G into Z is divisible by an integer m; this has an immediate application to a theorem in combinatorial topology.

2. Divisibility theorems, discrete groups. We use the following notations for a group X and an integer m.

$$mX = \text{all } mx, \qquad x \in X,$$
$$_mX = \text{all } x \in X \text{ such that } mx = 0.$$

Note that $x \in mX$ means that x is divisible by m.

In the theorems below, we use

$$H = \text{Hom}(G, Z). \tag{1}$$

Let $H'(G')$ denote all $h(g)$, $h \in H'$, $g \in G'$.

LEMMA 1. *Suppose G' is a subgroup of G,*

$$h \in H, \qquad h' \in \text{Hom}(G', Z).$$

Then h' can be extended over G so that $h = mh'$ in G' if and only if:

(α) *$h(G) \subset mZ$.*

(β) *$h'(mg) = h(g)$ if $mg \in G'$.*

(γ) *If m' is a divisor of m, then $h'(m'g) \in m'Z$ if $m'g \in G'$.*

The necessity of the conditions is clear. We prove the sufficiency first for the case

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