CONTINUED FRACTIONS AND BOUNDED ANALYTIC FUNCTIONS

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1. **Introduction.** In this paper we use the characterization given by Schur [6][1] for analytic functions bounded in the unit circle together with the Stieltjes integral representation of F. Riesz [5] for analytic functions with positive real parts, to obtain a new proof of a theorem [8] characterizing totally monotone sequences in terms of Stieltjes continued fractions. In the first place, Schur used an algorithm which he called a "continued fraction-like" algorithm. We begin by constructing from this an *actual* continued fraction algorithm, and we then characterize the class of analytic functions bounded in the unit circle in terms of this continued fraction. Next, we obtain by a simple transformation a continued fraction for functions with positive real parts.[2] This along with the above mentioned-theorem of F. Riesz leads to the theorem [8, pp. 165–166] that the sequence $\{c_p\}$ is totally monotone if and only if the power series $c_0 - c_1 z + c_2 z^2 - \cdots$ is the expansion of a continued fraction of the form

$$\frac{g_0}{1} + \frac{g_1 z}{1} + \frac{(1-g_1)g_2 z}{1} + \frac{(1-g_2)g_3 z}{1} + \cdots,$$

where $g_0 \geqq 0$, $0 \leqq g_p \leqq 1$, $p = 1, 2, 3, \cdots$.

2. **An actual continued fraction algorithm derived from the "continued fraction-like" algorithm of Schur.** The continued fraction which we shall consider is as follows:[3]

$$(2.1) \qquad \alpha_0 + \frac{(1-\alpha_0\bar{\alpha}_0)z}{\bar{\alpha}_0 z} \; \frac{1}{-\alpha_1} + \frac{(1-\alpha_1\bar{\alpha}_1)z}{\bar{\alpha}_1 z} \; \frac{1}{-\alpha_2} + \cdots,$$

in which the α_p are complex constants with moduli not exceeding unity, and z is a complex variable. It will be convenient to suppose that if for some p, $|\alpha_p| = 1$, then the continued fraction *terminates* with the first identically vanishing partial numerator.

The pth approximant of (2.1) will be denoted by $A_p(z)/B_p(z)$, where $A_0(z) = \alpha_0$, $B_0(z) = 1$, $A_1(z) = z$, $B_1(z) = \bar{\alpha}_0 z$, and the other nu-

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

[2] A special case of this was given in [9, p. 415].

[3] Hamel [2] used a somewhat different continued fraction for the purpose of characterizing analytic functions bounded in the unit circle. He was obliged to use an unconventional definition of convergence.