UNION-PRESERVING TRANSFORMATIONS OF SPACE

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1. General statement. Sophus Lie studied transformations from lineal-elements into lineal-elements, and also transformations from surface-elements into surface-elements of space. The contact group is obtained by requiring all unions to be converted into unions. Lie's fundamental theorems may be stated as follows. All the contact lineal-element transformations form the group of extended point transformations. The contact surface-element transformations which are not merely extended point transformations are defined completely by either a single directrix equation, or a pair of directrix equations. In the first case, a point corresponds to a surface; and in the second case, a point corresponds to a curve.[1]

We extend the preceding results by studying transformations in space from differential curve-elements of order n: $(x, y, z, y', z', \cdots, y^{(n)}, z^{(n)})$, where n is 2 or more, into lineal-elements (X, Y, Z, Y', Z'). An example of such a transformation arises in the problem of finding the locus of the centers of spherical curvature for an arbitrary space curve. This problem leads to a transformation from curve-elements of third order into lineal-elements.[2]

We determine the general class of union-preserving transformations by means of a directrix equation. Lie has obtained directrix equations only for contact transformations of surface-elements since there are no contact transformations of lineal-elements besides the extended point transformations. For a point-to-surface transformation, Lie's standard directrix equation is of the form $\Omega(X, Y, Z, x, y, z)=0$. For a point-to-curve transformation, there are two standard directrix equations of the forms $\Omega_1(X, Y, Z, x, y, z)=0$, $\Omega_2(X, Y, Z, x, y, z)=0$. We find that any general union-preserving transformation from curve-elements of order n into lineal-elements is completely determined by our *new directrix equation*, involving derivatives, $\Omega(X, Y, Z, x, y, z, y', z', \cdots, y^{(n-2)}, z^{(n-2)})=0$.

In the final part of our paper, we shall prove that the only available union-preserving transformations (in the whole domain of curve-

Presented to the Society, November 27, 1943; received by the editors August 17, 1943. We have studied the two-dimensional aspects of our new theory in Proc. Nat. Acad. Sci. (1943) and Revista des Matimaticas (1943). This leads to a large extension of the Huygens theory of evolutes and involutes and Lie theory.

[1] Lie-Scheffers, *Beruhrungstransformationen*.

[2] See Bull. Amer. Math. Soc. abstract 49-9-235 by Kasner and DeCicco, *A generalized theory of contact transformations*.