ON THE EXTENSION OF DIFFERENTIABLE FUNCTIONS

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The author has shown previously how to extend the definition of a function of class C^m defined in a closed set A so it will be of class C^m throughout space (see [1]).[1] Here we shall prove a uniformity property: If the function and its derivatives are sufficiently small in A, then they may be made small throughout space. Besides being bounded, we assume that A has the following property:

(P) There is a number ω such that any two points x and y of A are joined by an arc in A of length less than or equal to ωr_{xy} (r_{xy} being the distance between x and y).

This property was made use of in [2]; its necessity in the theorem is shown by two examples below.

A second theorem removes the boundedness condition in the first theorem, and weakens the hypothesis (P); its proof makes use of the proof of the first theorem. We remark that in each theorem, as in [1], the extended function is a linear functional of its values in A.

The proof of Theorem I is obtained by examining the proof in [1]; hence we assume that the reader has this paper before him, and we shall follow its notations closely.

THEOREM I. *Let A be a bounded closed set in n-space E with the property* (P), *and let m be a positive integer. Then there is a number α with the following property. Let $f(x)$ be any function of class C^m in A, with derivatives $f_k(x)$ $(\sigma_k = k_1 + \cdots + k_n \leqq m)$. Suppose*

$$|f_k(x)| < \eta \qquad (x \in A,\ \sigma_k \leqq m).$$

Then $f(x)$ may be extended throughout E so that

$$|f_k(x)| < \alpha\eta \qquad (x \in E,\ \sigma_k \leqq m).$$

Let d be the diameter of A, or 1 if this is larger, and let R be a spherical region of radius $2d$ with its center at a point of A. Set $f(x)=0$ in $E-R$. Then the extension of f in $R-A$ given in [1] will be shown to have the property, using

$$\alpha = 2n(m!)^n(m+1)^{3n}(433n^{1/2}d\omega)^m cN,$$

where N and c are as given in [1, §§11, 12]. Note that $433 = 4\cdot 108+1$.

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[1] Numbers in brackets refer to the references cited at the end of this paper.