TWO WORKS ON ITERATION AND RELATED QUESTIONS

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I want to give a brief account of results which have been communicated to me and which have been obtained by two young geometers, Eri Jabotinsky and Michel Lüntz. Mr. Lüntz is unhappily in a concentration camp in France. I must add that this is one of the reasons for the fact, for which I must apologize, that I am writing this exposé only now, although I have had both works in my possession for several months. The impossibility of communicating with Mr. Lüntz and therefore of asking him for any explanation made the examination of his paper especially difficult.

As is well known, the problem of iteration, a function $f(x)$ being given, consists in finding a one-parameter family of functions $f_n(x) = f(n,x)$ such that, for $n=1$, we have

$$f(1, x) = f(x) \tag{1}$$

and, moreover, for any m, n,

$$f_m[f_n(x)] = f_{m+n}(x); \tag{2}$$

also

$$f(0, x) = x \tag{3}$$

which follows from (2) as is seen by taking $m=0$.

The question is connected with Abel's functional equation

$$\phi[f(x)] = 1 + \phi(x) \tag{A}$$

because if ϕ is a solution of that equation a solution of (1), (2) will be given by

$$\phi[f_n(x)] = n + \phi(x). \tag{4}$$

Instead of (A), one can introduce Schroeder's equation

$$\psi[f(x)] = k\psi(x) \tag{A'}$$

(k a constant) in which ψ is connected with the unknown ϕ of (A) by $\psi = k^{\phi}$, and with the help of which the solution would be expressed by

$$\psi[(f_n(x)] = k^n\psi(x).$$

The fact that every solution of (A) gives a solution of the iteration

Presented to the Society, April 24, 1943, under the title *On fractional iteration and connected questions*; received by the editors July 1, 1943.