THE ADDITIVITY OF THE LEBESGUE AREA

J. W. T. YOUNGS

A triple of continuous functions T: $x^i(u^1, u^2)$, $i=1, 2, 3$, defined on a closed square Q $[0 \leqq u^1 \leqq 1,\ 0 \leqq u^2 \leqq 1]$ represents a surface $\mathfrak{S}$ (1.6, 1.17, 1.21).[1] If r is any closed rectangle in Q then we may speak of the triple T_r consisting of the above triple T with its range of definition restricted to r. This triple generates a surface $\mathfrak{S}(r)$. If r_1 and r_2 have no interior points in common, and $r_1+r_2=Q$, it is natural to hope that $L(\mathfrak{S})=L(\mathfrak{S}(r_1))+L(\mathfrak{S}(r_2))$ where L is the symbol used to indicate the Lebesgue area (3.13). This statement is certainly true whenever the Lebesgue area is given by the standard integral formula, since the Lebesgue integral is additive. However, in general it is false. It may be said that this note is concerned with the statement that if the triple is constant on $r_1 \cdot r_2$, the Lebesgue area is additive.

Stated in this fashion the theorem may appear to be new. Actually it is but a very special case of a fundamental problem in the theory of area. The classical conjecture is that the area is additive under the much weaker requirement that the triple be rectifiable on $r_1 \cdot r_2$. No proof of this conjecture has, to our knowledge, appeared. (In this connection see McShane [1, p. 138].)

In relation to existing literature the theorem of this note is included in a more general theorem due to Morrey [1]. In fact, the first case in Morrey's proof [1, p. 314] is essentially the theorem of this paper. A discrepancy in his argument was rectified by Radó and Reichelderfer [1] in an independently interesting discussion of stretching processes.

This result, therefore, is not new.

The point of this treatment is the utter simplicity of the stretching process here employed. The development serves to make a known result more accessible.

THEOREM 1. *If a polyhedron* $\mathfrak{P}$ (3.11, 1.21) *has a quasi-linear* (1.7) *representation* T: $x(u)$, $u \in Q$, *such that the image of the side* s $[u^1=1,\ 0 \leqq u^2 \leqq 1]$ *is interior to a sphere of diameter less than* ϵ *about the point* a $[a^1, a^2, a^3]$, *then there exists a polyhedron* $\mathfrak{P}^*$ *with quasi-linear representation* $T^*: x^*(u)$, $u \in Q$, *such that*

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[1] The notation and terminology of this note are largely due to Radó [1]; in fact, numbers in parentheses refer the reader to appropriate paragraphs in his paper. Numbers in brackets refer to the bibliography.