A CONVERGENCE THEOREM FOR LEBESGUE-STIELTJES INTEGRALS

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This paper deals with the following theorem.

THEOREM. *Let $g(x)$ be a bounded Borel measurable function defined everywhere on $(-\infty, \infty)$. Let $p_n(x)$ be a sequence of normalized functions,*[1] *$p_n \in V$, such that*

$$\sum_{n=1}^{\infty} V(p_n) < \infty. \tag{1}$$

Then $\sum_{n=1}^{\infty} p_n(x)$ is absolutely and uniformly convergent to a normalized function $p(x) \in V$, and

$$\sum_{n=1}^{\infty} \int_{-\infty}^{\infty} g(x)dp_n(x) = \int_{-\infty}^{\infty} g(x)dp(x). \tag{2}$$

We shall first prove this theorem in the following special cases:

(a) $g(x)$ is a bounded piecewise absolutely continuous function[2] in $(-\infty, \infty)$.

(b) $g(x)$ is continuous in a finite interval and vanishes identically outside this finite interval. (It need not necessarily be continuous at the end points of the interval.)

(c) $g(x)$ is a bounded continuous function in $(-\infty, \infty)$.

First let us prove our assertion concerning $p(x)$. Since $p_n(x)$ are normalized, we have[3] $\sum_{n=1}^{\infty} |p_n(x)| \ll \sum_{n=1}^{\infty} P_n \ll \sum_{n=1}^{\infty} V(p_n)$ where P_n is the upper bound of $|p_n(x)|$ for $-\infty < x < \infty$. Because of (1), it follows that the series $\sum_{n=1}^{\infty} p_n(x)$ is absolutely and uniformly convergent. Let $p(x)$ be the limit function. Evidently $p(x)$ is right-continuous and normalized. To show that $p(x) \in V$ it is sufficient to show that $p(x)$ is of bounded variation on $(-\infty, \infty)$. For $\xi > 0$ and any subdivision of $(-\xi, \xi)$, $-\xi = x_0 < x_1 < \cdots < x_{m-1} < x_m = \xi$, we

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[1] $p(x) \in V$ means that $p(x)$ is right-continuous and of bounded variation on the infinite interval $(-\infty, \infty)$. It is normalized if $p(0) = 0$. $V(p_n)$ denotes the total variation of p_n over $(-\infty, \infty)$.

[2] $f(x)$ is piecewise absolutely continuous in $(-\infty, \infty)$ if we can divide $(-\infty, \infty)$ into a finite number of intervals such that in each of these intervals $f(x)$ is absolutely continuous.

[3] "$<<$" is to be read "is dominated termwise by."