remarked that Theorem A may well carry, in such a study, a weight greater than that indicated by its relatively minor role in the proof of Theorem B.

UNIVERSITY OF MISSOURI

---

# THE EQUIVALENCE OF $n$-MEASURE AND LEBESGUE MEASURE IN $E_n$

ARTHUR SARD

Consider a set $A$ of points in euclidean $n$-space $E_n$. For each countable covering $\{A_i\}$ of $A$ by arbitrary sets consider the sum

$$\sigma = \sum_i c_m \delta(A_i)^m,$$

where $m$ is a fixed positive number, $c_m = \pi^{m/2}/2^m\Gamma[(m+2)/2]$, and $\delta(A)$ is the diameter of $A$. The constant $c_m$ is, for integral $m$, the $m$-volume of a sphere of unit diameter in $E_m$. Let $L_m(A;\alpha)$ be the greatest lower bound of all sums $\sigma$ corresponding to coverings for which $\delta(A_i) < \alpha$ for all $i$ $(\alpha > 0)$. We define the $m$-measure of $A$ as $L_m(A) = \lim_{\alpha\to 0} L_m(A;\alpha)$. We denote the outer Lebesgue measure of $A$ by $|A|$.

We shall show that *$n$-measure and outer Lebesgue measure are equal*: $L_n(A) = |A|$. A statement on this matter by W. Hurewicz and H. Wallman is true but misleading: these authors assert that $L_n(A)/c_n$ and $|A|$ may be unequal.[1]

F. Hausdorff has introduced an $m$-measure $L_m^S(A)$ defined as is $L_m(A)$ except that coverings by spheres are used instead of coverings by arbitrary sets. He has shown[2] that $L_n^S(A) = |A|$. However $L_m(A)$ and $L_m^S(A)$ are unequal in general, as A. S. Besicovitch has shown[3] for $m=1$, $n=2$. S. Saks[4] and others define $m$-measure as $L_m(A)/c_m$.

Our proof, which is an obvious extension of Hausdorff's proof, depends on two known theorems.

THEOREM I. *Of all sets in $E_n$ having a given diameter, the $n$-sphere has the greatest outer Lebesgue measure.*[5]

---

Received by the editors September 23, 1942, and, in revised form, April 2, 1943.

[1] W. Hurewicz and H. Wallman, *Dimension theory*, Princeton, 1941, p. 104.

[2] F. Hausdorff, *Dimension und äusseres Mass*, Math. Ann. vol. 79 (1919) p. 163.

[3] A. S. Besicovitch, *On the fundamental geometrical properties of linearly measurable plane sets of points*, Math. Ann. vol. 98 (1928) pp. 458–464. R. L. Jeffery, *Sets of k-extent in n-dimensional space*, Trans. Amer. Math. Soc. vol. 35 (1933) p. 634.

[4] S. Saks, *Theory of the integral*, Warsaw, 1937, pp. 53–54.