THE RADICAL OF A GROUP WITH OPERATORS

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In a recent note (*The radical of a non-associative algebra*, Bull. Amer. Math. Soc. vol. 48 (1942) pp. 893–897) A. A. Albert has defined the radical of a general linear algebra, and deduced some of its properties from his theory of isotopy. The purpose of the present note is to extend this concept to groups with operators, and to derive similar properties from lattice theory.

Let G be a group with a class Ω of endomorphisms including all inner automorphisms.[1] By an *ideal*, we mean a subgroup S of G such that $s \in S$ and $w \in \Omega$ imply $sw \in S$. Clearly G and the group identity 0 are ideals; any other ideal is called a *proper ideal.* We recall that the ideals of G form a *modular lattice*;[2] we shall assume below that this has *finite length.*

We define G to be *prime* if and only if it has no proper ideals; it is well known (and easy to prove) that G/S is prime if and only if S is maximal. Now let $\mathfrak{Z}$ be *any* class of groups with operators (specializing to zero algebras) which is invariant under isomorphisms,[3] and let us define a *simple* group (with operators) to be any prime group not in the exceptional class $\mathfrak{Z}$. We define a direct sum of prime groups to be *semiprime*, and (following Albert and others) a direct sum of simple groups to be *semisimple.*

We further define the *ϕ-ideal* Φ of G as the intersection of all maximal ideals (by analogy with the ϕ-subgroup of a group), the *radical* R of G as the intersection of all maximal ideals S such that G/S is simple, and denote by Z the intersection of all maximal ideals T such that $(G/T) \in \mathfrak{Z}$.

THEOREM 1. *The quotient group G/S is semiprime if and only if S contains Φ.*

PROOF. If G/S is semiprime, then it can be written as a direct sum $G/S = (S_1/S) + \cdots + (S_r/S)$, where the (S_k/S) are prime. The $B_i = S_1 \cup \cdots \cup S_{i-1} \cup S_{i+1} \cup \cdots \cup S_r$ are maximal ideals; and

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[1] If G is commutative, Ω may be void. The case of a linear algebra is included by requiring Ω to consist of all scalar multiplications $x \rightarrow \lambda x$, all left multiplications $x \rightarrow ax$, and all right multiplications $x \rightarrow xa$. It follows that ideals and direct sums correspond.

[2] In the sense of the author's *Lattice theory*, Amer. Math. Soc. Colloquium Publications, vol. 25, New York, 1940. This will be referred to below as [LT].

[3] It is to be observed that a zero algebra is *not* a "zero" group with operators: $\lambda x \neq 0$ for scalar operators λ.