UNIFORM CONVEXITY. III

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It is the purpose of this note to fill out certain results given in two recent papers on uniform convexity of normed vector spaces.[1] A normed vector space[2] B is called *uniformly convex* with *modulus of convexity* δ if for each $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that for every two points b and b' of B satisfying the conditions $\|b\| = \|b'\| = 1$ and $\|b - b'\| \geqq \epsilon$ the quantity $\|b + b'\| \leqq 2(1 - \delta(\epsilon))$. If $\|b_0\| = 1$, B is said to be *locally uniformly convex near* b_0 if there is a sphere about b_0 in which the condition for uniform convexity holds. Theorem 1 shows that all properties of normed vector spaces which are invariant under isomorphism are the same for uniformly convex and locally uniformly convex spaces. Theorem 2 gives a necessary condition for isomorphism with a uniformly convex space. The condition is in terms of isomorphisms of finite dimensional subspaces and is suggested by examples given in [I]; it is not known whether the condition is sufficient. Theorem 3 is somewhat more general than Theorem 3 of [II]; it uses uniformly convex function spaces instead of the l_p spaces of [II].

A *cone* C in B is a set which contains all of every half line from the origin through each point of C.

LEMMA 1. *A normed vector space B is locally uniformly convex near b_0 if and only if there exists a convex cone C, with b_0 in its interior, such that for every ϵ there is a $\delta_1(\epsilon) > 0$ such that the conditions $\|b\| \leqq 1$, $\|b'\| \leqq 1$, and $\|b - b'\| \geqq \epsilon$ imply $\|b + b'\| \leqq 2(1 - \delta_1(\epsilon))$ for every pair of points b and b' in C.*

If this condition is satisfied there is obviously a sphere about b_0 inside C, so that in that sphere $\delta(\epsilon)$ can be taken equal to $\delta_1(\epsilon)$. On the other hand, if there is a sphere of radius $2k$ about b_0 in which δ can be defined, it can be shown that it suffices to let C be the cone through points of the sphere of radius k about b_0 and to let $\delta_1(\epsilon) = \inf [\epsilon/10, \delta(4\epsilon/5)/2]$.

LEMMA 2. *If the cone C of Lemma 1 contains a sphere about b_0 of radius k, if $\|b\| \leqq 1$ and if $\|b - b_0\| \geqq k$, then $\|b + b_0\| < 2 - \delta_1(k)$.*

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[1] These papers are [I] *Reflexive Banach spaces not isomorphic to uniformly convex spaces*, Bull. Amer. Math. Soc. vol. 47 (1941) pp. 313–317, and [II] *Some more uniformly convex spaces*, Bull. Amer. Math. Soc. vol. 47 (1941) pp. 504–507.

[2] See Banach, *Théorie des opérations linéaires*, Warsaw, 1932, for general definitions.