EXTENSIONS OF CERTAIN DYNAMICAL THEOREMS OF HALPHEN AND KASNER

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1. **Introduction.** We wish to present here some extensions of certain theorems of Halphen and Kasner concerning the dynamical trajectories of positional fields of force in space.

Kasner has developed the differential geometry of the dynamical trajectories of general positional fields of force in the plane and in space in his Princeton Colloquium Lectures.[1] Recently Kasner and the author have introduced the concept of *generalized* fields of force which depend not only upon the position of the point but also upon the direction through the point.[2] In a generalized field of force in space, there are ∞^5 dynamical trajectories just as in the ordinary positional case.

The theorem of Halphen which we wish to extend to generalized fields of force is that all positional fields of force whose ∞^5 dynamical trajectories are plane curves are of the central or parallel type. That is, the lines of force are all straight lines through a fixed point O (which may be finite or at infinity). Moreover the ∞^5 trajectories consist of the ∞^2 systems of ∞^3 ordinary plane dynamical trajectories of the central or parallel type, each such system lying in a plane through the point O. There are $\infty^{3+f(3)}$ such systems of ∞^5 dynamical trajectories,[3] and the number of central fields of force is $\infty^{4+f(3)}$.

We find that all generalized fields of force whose ∞^5 dynamical trajectories are plane curves form a more extensive class. The ∞^5 trajectories consist, in general, of ∞^2 systems of ∞^3 generalized plane dynamical trajectories, each such system lying in a plane tangent to a base surface Σ. The various degenerate situations are four in number and may be described as follows: (i) The base planes are tangent to a given curve C, (ii) the base planes pass through a given finite point O, (iii) the base planes are all parallel to the tangent planes of a given

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[1] Kasner, *Differential geometric aspects of dynamics*, Amer. Math. Soc. Colloquium Publications, vol. 3, 1913, 1934. Also see Trans. Amer. Math. Soc. vols. 7–11 (1906–1910).

[2] See (1) *A generalized theory of dynamical trajectories*, Trans. Amer. Math. Soc. vol. 54 (1943) pp. 23–38; and (2) *Generalized dynamical trajectories in space* abstract 49-3-120.

[3] The symbol $\infty^{kf(n)}$ denotes the content of a geometric manifold which depends on k functions of n variables only. See Kasner, *A notation for infinite manifolds*, Amer. Math. Monthly vol. 49 (1942) pp. 243–244.