RANDOM SUMMABILITY AND FOURIER SERIES

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Fejér's theorem that the first arithmetic mean of the partial sums of the Fourier series of a function converges to the function p.p. is equivalent to saying the average values of the partial sums, taken in order, are close to the function. It is interesting to ask whether, if the partial sums are chosen *at random*, and then averaged, the new averages will be close to the function. We shall show that this is true *most of the time*.

Let us define *random summability* in terms of the existence of the limit

$$\lim_{n\to\infty} \frac{\sum_1^n s_k(x)(1 + r_k(t))}{\sum_1^n (1 + r_k(t))}$$

where $r_k(t)$ are the Rademacher functions, $r_k(t) = \text{sign} \sin 2^{k+1}\pi t$, and $s_k(x)$ are the partial sums of $\sigma(f) = a_0 + \sum_1^\infty (a_k \cos kx + b_k \sin kx)$.

The only fact we require concerning the Rademacher functions is [1][1]: If $\sum_1^\infty a_k^2 < \infty$, $\sum_1^\infty a_k r_k(t)$ converges p.p. in t.

Now we have

$$\frac{\sum_1^n s_k(x)(1 + r_k(t))}{\sum_1^n (1 + r_k(t))} = \frac{\sum_1^n s_k(x)}{n} \cdot \frac{n}{\sum_1^n (1 + r_k(t))} + \frac{\sum_1^n s_k(x) r_k(t)}{n} \cdot \frac{n}{\sum_1^n (1 + r_k(t))}.$$

Also $\sum_1^\infty r_k(t)/k^{1/2} \log (k+1)$ converges p.p. in t, by the above, since $\sum_1^\infty 1/k \log (k+1)^2 < \infty$, so that $\sum_1^n r_k(t) = o(n^{1/2} \log n)$, by Kronecker's Lemma (much more is known, but this is more than needed). Hence, certainly $\sum_1^n r_k(t) = o(n)$, and $\sum_1^n (1 + r_k(t)) \sim n$.

By Fejér's theorem $n^{-1} \sum_1^n s_k(x) \to f(x)$ p.p. in x, so that it remains to prove $n^{-1} \sum_1^n s_k(x) r_k(t) \to 0$ p.p. in x and t.

But $\sum_1^\infty (s_k(x)/k)^2 < \infty$ p.p. in x, since $s_k(x) = o(\log k)$ p.p. in x [2]. Therefore $\sum_1^\infty s_k(x) r_k(t)/k$ converges p.p. in x and t, and using Kronecker's Lemma again $\sum_1^n s_k(x) r_k(t) = o(n)$.

Thus, we have proved: If $f(x)$ belongs to L, the Fourier series of $f(x)$ is *random summable* p.p. in x, *most of the time*.

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[1] Numbers in brackets indicate references at end of paper.