TWO NOTES ON MEASURE THEORY

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I. In a recent paper [1],[1] Saks has indicated a construction whereby a Carathéodory outer measure can be produced on any compact metric space M, provided that a certain linear functional Φ is defined on the set $\mathfrak{C}$ of all continuous real-valued functions whose domain is M. The functional Φ is required to be non-negative for non-negative functions, and to have the property that if the sequence $\{f_n\}$ has the uniform limit 0, then the sequence $\Phi(f_n)$ is a null-sequence. (The measure itself can be defined without this last property.) The purpose of this note is to show that such a linear functional always exists, in a non-trivial form, specifically, so that $\Phi(1)=1$.

We consider the set $\mathfrak{C}$ as a linear space, and together with $\mathfrak{C}$ the linear space $\mathfrak{K}\subset\mathfrak{C}$, where $\mathfrak{K}$ consists of all constant functions. On the entire space $\mathfrak{C}$, we define a functional $p(f)=\sup_{x\in M} f(x)$. This least upper bound always exists, since M, being a compact metric space, is a bicompact space, on which every continuous real-valued function is bounded. It is easy to verify that $p(f+g)\leqq p(f)+p(g)$, for all $f, g\in\mathfrak{C}$, and that $p(tf)=tp(f)$ whenever t is a non-negative real number. We define a linear functional Φ on the subspace $\mathfrak{K}$ as follows: $\Phi(f)=f(x)$ for an arbitrary $x\in M$. It is clear that $\Phi(f)=p(f)$ for $f\in\mathfrak{K}$ and that Φ is linear on $\mathfrak{K}$. By virtue of the celebrated theorem of Hahn-Banach, it appears that Φ can be extended linearly to all of $\mathfrak{C}$ in such a fashion that $\Phi(f)\leqq p(f)$ for all $f\in\mathfrak{C}$. We further observe that Φ may be taken non-negative for non-negative functions. For, if Φ has been defined by the Hahn-Banach construction for all $f\in\mathfrak{B}$, where $\mathfrak{K}\subset\mathfrak{B}\subset\mathfrak{C}$, $\mathfrak{B}\neq\mathfrak{C}$, and if $g\in\mathfrak{C}-\mathfrak{B}$ and $g\geqq 0$, then the number $a=\inf_{f\in\mathfrak{B}}(p(f+g)-\Phi(f))$ is an upper bound to possible values for $\Phi(g)$. a, however, is plainly non-negative, so that $\Phi(g)$ may always be taken non-negative. Suppose now that the sequence of functions $\{f_n\}$ has the uniform limit 0. The function $\epsilon-f_n$ is non-negative for all $n>N(\epsilon)$, $N(\epsilon)$ being some natural number dependent upon the arbitrary positive number ϵ. Accordingly, $\Phi(\epsilon-f_n)=\Phi(\epsilon)-\Phi(f_n)=\epsilon\Phi(1)-\Phi(f_n)=\epsilon-\Phi(f_n)\geqq 0$. Likewise, it is easy to show that $\epsilon+\Phi(f_n)\geqq 0$ for all sufficiently large n. It follows at once that $\lim_{n\to\infty}\Phi(f_n)=0$. It is proved in Saks [1] that the functional Φ can

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[1] Numbers in brackets refer to correspondingly numbered articles in the bibliography at the end of the paper.