RADICAL EXTENSIONS AND CROSSED CHARACTERS

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E. Witt[1] has given a theory of abelian extensions of fields containing sufficiently many roots of unity which consists essentially, as has been remarked before,[2] in applying the theory of characters of finite abelian groups. There exists now a sufficiently developed theory of crossed characters,[3] and it is the object of this note to show that a fairly complete and simple theory of radical extensions may be obtained if one follows Witt's treatment[4] of abelian extensions, only substituting for the classical theory of characters the theory of crossed characters.[5]

Suppose that the commutative field[6] K is a finite, normal, and separable extension of the field F, that the characteristic of the field K is either 0 or prime to the given integer m, and that E is the group of the mth roots of unity contained[7] in K. The Galois group G of the extension K of F consists of all the F-automorphisms of the field K (automorphisms of the field K which leave the elements in F invariant). Every automorphism g in G induces in E an automorphism which we also denote by g, and the correspondence mapping the automorphism g in G upon the automorphism g of E shall be denoted by C.

A C-character of the group G is a single-valued G to E function $f(g)$, satisfying the functional equation $f(u)^v f(v) = f(uv)$.

LEMMA 1. *The function v^{1-g} of the element g in G, for v an element in K, is a C-character of G if, and only if, v^m is an element, not 0, in F.*

PROOF. If v^{1-g} is a C-character of G, then v^{1-g} is for every g in G an element in E, so that $(v^m)^{1-g} = 1$ for every g in G. Thus v^m is a fixed element of the Galois group of K over F, proving that v^m is an element, not 0, in F.

Assume conversely that $v^m \neq 0$ is in F. Then $(v^{1-g})^m = (v^m)^{1-g} = 1$ for

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[1] Witt [1], [2]; the references refer to the bibliography at the end of the paper.

[2] Baer [1].

[3] Baer [2].

[4] Or the treatment as suggested in Baer [1].

[5] The importance of the theory of crossed characters for the theory of radical extensions has recently been stressed by MacLane-Schilling [1].

[6] As all the fields will be commutative, we shall omit the word "commutative" in the future.

[7] K need not contain m distinct mth roots of unity; cp. Theorem 3 below.