EXACT nTH DERIVATIVES

HOWARD LEVI

Let y be a function of x with derivatives of all orders, and let θ be a function of x, y, and a finite number of derivatives of y. If, independently of the choice of the function y, θ is the nth total derivative of some function ψ of x, y, and derivatives of y, then we shall call θ an *exact nth derivative*. The problem with which this note is concerned is to determine, for any given function θ and positive integer n, if θ is an exact nth derivative. The case for which $n=1$ is completely covered by the well known Euler differential equation which arises in the simplest problem of the calculus of variations. For a function θ to be an exact first derivative, it is necessary and sufficient that θ satisfy the Euler differential equation. The contribution of this paper is the treatment of the cases in which n exceeds unity. A system of n differential equations is developed, satisfaction of which by θ constitutes a necessary condition that θ be an exact nth derivative. These equations do not yield an altogether satisfactory sufficient condition. It turns out that if θ satisfies the equations in question, it may still fail to be an exact nth derivative. However, under these circumstances, θ must differ from an exact nth derivative by a function of very special character.

Notation. Let us suppose y to be an arbitrary function of x possessing derivatives of all orders. We shall denote the jth derivative of y with respect to x by y_j, and sometimes denote y itself by y_0. We suppose θ to be a function of x, y, and of finitely many of the y_j, possessing partial derivatives of all orders with respect to all its arguments. The operation of differentiation with respect to x will be indicated by the symbol D; thus $D=\partial/\partial x+\sum y_{i+1}\partial/\partial y_i$. We shall understand that the range of the subscript i in D extends from zero to plus infinity, recognizing that when D operates on a function of x, y, and of finitely many of the y_j it reduces to a finite sum. The symbol D^i, where i is a positive integer, will denote the operation of taking the ith derivative. We shall use the expression $C_{p,q}$ to denote the binomial coefficient $p\cdot(p-1)\cdots(p-q+1)/q!$ where q is a nonnegative integer and p is any integer.

Summary of results. Let n be a positive integer. Let operators E_t, $t=1,\cdots,n$, be defined as follows. Expand, formally, $E_t=(1+D\partial/\partial y_1)^{-t}\partial/\partial y$ as the product by $\partial/\partial y$ of a power series in $D\partial/\partial y_1$, and replace terms $(D\partial/\partial y_1)^i\partial/\partial y$ by $D^i\partial/\partial y_i$. *Let there be a*

Received by the editors January 15, 1943.