ON AUTOMORPHISMS OF COMPACT GROUPS

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1. Introduction and definitions. Let G be a compact abelian group and α a continuous automorphism of G. We write G multiplicatively and use, accordingly, the exponent notation for automorphisms. Thus the image under α of the element $x \in G$ will be denoted by x^α; similarly we shall write for (complex valued) functions $f(x)$, $f^\alpha(x) = f(x^\alpha)$.[1]

If m is Haar measure[2] in G (normalized so that $m(G) = 1$) we consider the set function $m'(E) = m(E^\alpha)$. (E^α is the set of all x^α, $x \in E$.) Since m' is a measure on G possessing all defining properties of m it follows from the uniqueness of Haar measure[3] that $m'(E) = m(E)$ for every measurable set E. In other words α is a measure preserving transformation of G; the purpose of this note is to investigate a few simple properties of α from the point of view of measure theory.

We shall make use of the Pontrjagin duality theory,[4] and, in particular, we shall need the fact that the group of automorphisms of G is essentially the same as that of the character group G^*. More precisely: if to any $\phi = \phi(x) \in G^*$ we make correspond $\phi^\alpha = \phi^\alpha(x) = \phi(x^\alpha)$, then $\phi^\alpha \in G^*$, and the correspondence $\phi \rightarrow \phi^\alpha$ is an automorphism of G^*. The duality theory also enables us to reverse this argument: every automorphism of G^* is induced in this way by a continuous automorphism of G.

We recall some standard definitions from ergodic theory. A measure preserving transformation α (not necessarily an automorphism) is *ergodic* if the only (complex valued, measurable) solutions f of the equation $f^\alpha = f$ are constant almost everywhere. The transformation α is *mixing* if the only (complex valued, measurable) solutions f of the equation $f^\alpha = \lambda f$, for any constant λ, are constant almost everywhere.[5] (It is true, though irrelevant, that for $\lambda \neq 1$ even a constant fails to be a solution unless it is zero.) It is well known that the mapping

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[1] This notation dovetails, as usual, with ordinary exponentiation in G; thus $x^{3\alpha^2} = (x^3)^{\alpha^2} = (x^{\alpha^2})^3$, and so on.

[2] For a general discussion of measure theory in topological groups see A. Weil, *L'intégration dans les groupes topologiques et ses applications*, Paris, 1938.

[3] Weil, op. cit., pp. 36–38.

[4] Weil, op. cit., chap. 6.

[5] See E. Hopf, *Ergodentheorie*, Berlin, 1937, chap. 3, for a discussion of the fact that these definitions are equivalent to the ones more commonly given.