ON LINEAR SPACES WHICH MAY BE RENDERED COMPLETE NORMED METRIC SPACES

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In this paper, we obtain a characterization of linear spaces which may be normed so as to become complete, linear, normed metric spaces. In this connection, K. Kunugui[1] and M. Fréchet[2] have shown that every metric space S is isometric with a subset of a complete, linear, normed metric space. It follows from our result that if the cardinal number of S is the limit of a denumerable sequence of cardinals, then there is no complete, linear, normed metric space isometric with S. Results on topological spaces which may be rendered linear, normed metric spaces and complete, linear, normed metric spaces have been given by A. Kolmogoroff[3] and B. Z. Vulich.[4]

It will be assumed that the reader is familiar with certain elementary portions of the theory of linear and metric spaces, and with transfinite cardinal and ordinal numbers.[5] Using the generalized continuum hypothesis and normal order theorem, we prove the following:

THEOREM. *A necessary and sufficient condition that a linear space may be made a complete, linear, normed metric space by a suitable definition of norm is that the cardinal number of its Hamel basis should not be the limit of any denumerable sequence of cardinals which precede it.*

A Hamel basis of a linear space S is a subset T of S such that every element of S is a linear combination, with real coefficients, of a finite number of elements of T, and there is no proper subset of T with this property. The following properties of a Hamel basis will be used in demonstrating the theorem, and are given without proof:

(a) A linear space S has a Hamel basis.[6]

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[1] K. Kunugui, *Applications des espaces à une infinité de dimensions à la théorie des ensembles*, Proc. Imp. Acad. Tokyo vol. 11 no. 9 pp. 351–353.

[2] M. Fréchet, *Sur les espaces distanciés*. Memorial volume dedicated to D. A. Grave. Moscow, 1940, pp. 265–267.

[3] A. Kolmogoroff, *Zur Normierbarkeit eines allgemeinen topologischen Raumes*, Studia Mathematica, vol. 5 (1934) pp. 29–33.

[4] B. Z. Vulich, *Linear spaces with given convergence*, Leningrad State University Annals, Mathematics Series vol. 10 (1940) pp. 40–63.

[5] For the elements of transfinite number theory, see F. Hausdorff, *Mengenlehre*, Berlin, 1927; for the theory of linear spaces, see S. Banach, *Théorie des opérations linéaires*, Warsaw, 1932.

[6] Proofs of (a) and (b) have been given by H. Löwig, *Über die Dimension linearer Räume*, Studia Mathematica, vol. 5 (1934) pp. 18–24.