ON EXTENSION OF WRONSKIAN MATRICES

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1. **Introduction.** By an interval J we shall understand a finite interval of the type $a \leqq x \leqq b$. If $u_1(x), u_2(x), \cdots, u_n(x)$ are real functions possessing finite derivatives of the first t orders in an interval J and $0 \leqq s \leqq t$, we call the functional matrix

$$M_s(u_1, \cdots, u_n) \equiv \begin{Vmatrix} u_1 & u_2 & \cdots & u_n \\ u_1' & u_2' & \cdots & u_n' \\ \cdot & \cdot & \cdots & \cdot \\ u_1^{(s)} & u_2^{(s)} & \cdots & u_n^{(s)} \end{Vmatrix}$$

their Wronskian matrix of order s. The Wronskian $W(u_1, \cdots, u_n)$ is the determinant of the matrix $M_{n-1}(u_1, \cdots, u_n)$.

The principal result we obtain is that if $n \leqq s \leqq t$ and the Wronskian matrix of order s for n arbitrary functions $u_1(x), u_2(x), \cdots, u_n(x)$ of class[1] $C^{(t)}$ in an interval J has constant rank n, there exists a function $u_{n+1}(x)$ of class $C^{(t)}$ such that the extended matrix $M_s(u_1, \cdots, u_{n+1})$ has constant rank $n+1$ in J. We employ a theorem of Curtiss[2] which may be stated in the form:

THEOREM C. *If $u_1(x), u_2(x), \cdots, u_n(x)$ are functions of class $C^{(t)}$ in an interval J and their Wronskian matrix of order t has rank n throughout J, then the Wronskian $W(u_1, \cdots, u_n)$ has at most isolated zeros.*

From the extension property of Wronskian matrices we obtain a sufficient condition, in terms of the rank of a certain functional matrix, that an arbitrary set of functions having suitable class properties be solutions of an ordinary homogeneous linear differential equation.

2. **Lemmas.** We first prove two lemmas.

LEMMA 1. *If $\delta, c_1, c_2, \cdots, c_n$ are given constants with $\delta > 0$, there exists a function $f(x)$ of class $C^{(n)}$ in the interval $-1 \leqq x \leqq 1$ which satisfies the conditions*: (1) $|f(x)| \leqq \delta$, $-1 \leqq x \leqq 1$; (2) $f^{(i)}(-1) = f^{(i)}(1) = 0$, $i = 0, 1, \cdots, n$; (3) $f(0) = 0$, $f^{(i)}(0) = c_i$, $i = 1, 2, \cdots, n$.

Presented to the Society, September 10, 1942; received by the editors August 25, 1942.

[1] Each function has continuous derivatives of the first t orders at every point of J.

[2] D. R. Curtiss, *The vanishing of the Wronskian and the problem of linear dependence*, Math. Ann. vol. 65 (1908) Theorem 4.