A CONJECTURE OF ORE ON CHAINS IN PARTIALLY ORDERED SETS

SAUNDERS MACLANE

In a recent investigation, Ore[1] has given a form of the Jordan-Hölder theorem valid for an arbitrary partially ordered set P. This theorem involves essentially the deformation of one chain into another by successive steps, each step being like that used in the conventional Jordan-Hölder theorem. Ore observes that his first theorem would be slightly easier to apply if it were proved under a weaker hypothesis. The modified theorem runs as follows:[2]

THEOREM. *If P is a partially ordered set in which every chain joining two elements is finite, then any complete chain between two elements $b<a$ can be deformed into any other complete chain between the same two elements.*

The proof rests on this lemma:

LEMMA. *Under the hypothesis of the theorem, if C is a complete chain from b to a which cannot be deformed into the complete chain D from b to a, there exist in P elements $b'<a'$ and complete chains C' and D' from b' to a' such that C' cannot be deformed into D' and such that $b \leqq b'$, $a' \leqq a$ where either $b<b'$ or $a'<a$.*

PROOF. *Case* 1. C and D have in common the element e, $b<e<a$. Then either C_b^e cannot be deformed into D_b^e, or C_e^a cannot be deformed into D_e^a. In these two cases, set $b'=b$, $a'=e$ or $b'=e$, $a'=a$, respectively.

Case 2. C and D have no elements in common. Since C cannot be deformed into D, they cannot together constitute a simple cycle. There will then exist, say, elements c in C and d in D with $b<c<a$, $b<d<a$ and an element m in P with $c \leqq m<a$, $d \leqq m<a$. Because of the hypothesis that every chain in P joining two elements is finite, there will exist in P finite complete chains E_m^a, F_c^m, G_d^m. Then b is joined to a by four complete chains,

$$C_b^c + C_c^a, \qquad C_b^c + F_c^m + E_m^a,$$
$$D_b^d + G_d^m + E_m^a, \qquad D_b^d + D_d^a.$$

Received by the editors December 15, 1942.

[1] Oystein Ore, *Chains in partially ordered sets*, Bull. Amer. Math. Soc. vol. 49 (1943) pp. 558–566.

[2] Terminology and notation follow the paper of Ore.