ALTERNATIVE ALGEBRAS OVER AN ARBITRARY FIELD

R. D. SCHAFER[1]

The results of M. Zorn concerning alternative algebras[2] are incomplete over modular fields since, in his study of alternative division algebras, Zorn restricted the characteristic of the base field to be not two or three. In this paper we present first a unified treatment of alternative division algebras which, together with Zorn's results, permits us to state that any alternative, but not associative, algebra A over an arbitrary field F is central simple (that is, simple for all scalar extensions) if and only if A is a Cayley-Dickson algebra[3] over F.

A. A. Albert in a recent paper, *Non-associative algebras* I: *Fundamental concepts and isotopy*,[4] introduced the concept of isotopy for the study of non-associative algebras. We present in the concluding section of this paper theorems concerning isotopes (with unity quantities) of alternative algebras. The reader is referred to Albert's paper, moreover, for definitions and explanations of notations which appear there and which, in the interests of brevity, have been omitted from this paper.

1. **Alternative algebras.** A distributive algebra A is called an *alternative algebra* if $ax^2=(ax)x$ and $x^2a=x(xa)$ for all elements a, x in A. That is, in terms of the so-called right and left multiplications, A is alternative if $R_{x^2}=(R_x)^2$ and $L_{x^2}=(L_x)^2$.

The following lemma, due to R. Moufang,[5] and the Theorem of Artin are well known.

LEMMA 1. *The relations* $L_xR_xR_y=R_{xy}L_x$, $R_xL_xL_y=L_{yx}R_x$, *and* $R_xL_{xy}=L_yL_xR_x$ *hold for all* a, x, y *in an alternative algebra* A.

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[2] M. Zorn, *Theorie der Alternativen Ringe*, Abh. Math. Sem. Hamburgischen Univ. vol. 8 (1930) pp. 123–147; *Alternativkörper und Quadratische Systeme*, ibid. vol. 9 (1933) pp. 395–402. See M. Zorn, *Alternative rings and related questions* I: *Existence of the radical*, Ann. of Math. (2) vol. 42 (1941) pp. 676–686, to verify that the Peirce decomposition in Zorn's first paper is valid for F of characteristic two.

[3] Defined over an arbitrary field by A. A. Albert in his *Quadratic forms permitting composition*, Ann. of Math. (2) vol. 43 (1942) pp. 161–177.

[4] Ann. of Math. (2) vol. 43 (1942) pp. 685–708.

[5] R. Moufang, *Zur Struktur von Alternativkörpern*, Math. Ann. vol. 110 (1934) pp. 416–430.