THE NUMBER OF INDEPENDENT COMPONENTS OF THE TENSORS OF GIVEN SYMMETRY TYPE

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Let $T_{i_1 \cdots i_p}$ be an arbitrary covariant tensor with respect to an n-dimensional coordinate system, and let

$$(1) \qquad T_{i_1\cdots i_p} = {}_{[p]}T_{i_1\cdots i_p} + \cdots + {}_{[\alpha]}T_{i_1\cdots i_p} + \cdots + {}_{[1^p]}T_{i_1\cdots i_p}$$

represent the decomposition[1,2] of $T_{i_1 \cdots i_p}$ into tensors of various symmetry types, the tensor ${}_{[\alpha]}T_{i_1 \cdots i_p}$ corresponding to the partition $[\alpha]$ of the indices $i_1 \cdots i_p$. The number of independent (scalar) components of $T_{i_1 \cdots i_p}$ is n^p; and if c_α denotes the number of components of ${}_{[\alpha]}T_{i_1 \cdots i_p}$, then

$$(2) \qquad n^p = c_{[p]} + \cdots + c_{[\alpha]} + \cdots + c_{[1^p]} = \sum c_\alpha.$$

For $p=2, 3, 4$, J. A. Schouten[3] has obtained expressions for the c_α's in terms of n; but the difficulties of his method become great for larger values of p. The purpose of this paper is to present a method of obtaining c_α in terms of n from the character table for the symmetric group on p letters.

Associated with the immanant tensor[2] $I^{(i)}_{(j)} \equiv {}_{[\alpha]}I^{i_1\cdots i_p}_{j_1\cdots j_p}$ we have defined the numerical invariant $r=r_\alpha$, the rank[4] of $I^{(i)}_{(j)}$, which is the greatest integer r for which the tensor

$$(3) \qquad I^{(i_1)\cdots(i_r)}_{(j_1)\cdots(j_r)} = \begin{vmatrix} I^{(i_1)}_{(j_1)} & \cdots & I^{(i_1)}_{(j_r)} \\ \cdot & & \cdots \\ I^{(i_r)}_{(j_1)} & \cdots & I^{(i_r)}_{(j_r)} \end{vmatrix}$$

does not vanish; here $(i_\lambda) = i_{\lambda 1} \cdots i_{\lambda p}$. For convenience, let us regard $I^{(i)}_{(j)}$, for each (i), as a vector $V_{(j)}$ in $N=n^r$ dimensions. Then from the above definition, it is clear that exactly r_α of the N vectors $V_{(j)}$ are linearly independent. Since ${}_{[\alpha]}T_{(j)} \equiv {}_{[\alpha]}T_{j_1 \cdots j_p}$ may be defined by

$$(4) \qquad {}_{[\alpha]}T_{(j)} = {}_{[\alpha]}I^{(l)}_{(j)}T_{(l)};$$

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[1] H. Weyl, *The classical groups*, Princeton, 1939, chap. IV.

[2] T. L. Wade, *Tensor algebra and Young's symmetry operators*, Amer. J. Math. vol. 63 (1941) pp. 645–657.

[3] J. A. Schouten, *Der Ricci-Kalkul*, Berlin, 1924, chap. VII.

[4] Richard H. Bruck and T. L. Wade, *Bisymmetric tensor algebra*, II, Amer. J. Math. vol. 64 (1942) pp. 734–753. We shall refer to this paper as B.T.A.II.