ON THE CROSSING OF EXTREMALS AT FOCAL POINTS

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Morse and Littauer[1] have proved the following theorem for an analytic Finsler space, where g is an extremal transversal to the (analytic) hypersurface Σ.

THEOREM. *A necessary and sufficient condition that p on g be a focal point of Σ is that the family of extremals cut transversaly by Σ near g shall fail to cover the neighborhood of p simply.*

The purpose of the present paper is to prove this theorem on the weaker hypothesis that the Finsler space and Σ are of class C'''.

As pointed out in M. L. the sufficiency of the condition is trivial, and in proving the necessity there is no loss in generality if we assume p to be a first focal point. It is further clear from M. L. that the theorem is a consequence of the following lemma.

LEMMA I. *If p is a first focal point on g contained in a (simply covering) field R of extremals transversal to Σ, then there exists a first focal point q covered by R and a subfield S of R covering q and such that the Hilbert integral is independent of path for paths confined to S.*

Before proceeding to the proof of Lemma I we will establish a secondary lemma.

LEMMA II. *Let T be a transformation of class C' mapping a closed coordinate neighborhood A into a closed Riemannian manifold B, then almost all points of B (in the measure theoretic sense) have finite counter images.*

PROOF. Call the set of points $K\subset A$ at which the Jacobian of T vanishes critical points, then I assert that if the counter image $T^{-1}b$, $b\in B$, is infinite, it contains a critical point. In fact if b^i are the coordinates of such a point b, there is a convergent sequence of points of $T^{-1}b$ with coordinates a^i_σ approaching a point a_0 from a definite direction, as is expressed by the following set of equations.

$$\begin{gathered} \overset{i}{a}_\sigma \to \overset{i}{a}_0,\ \overset{i}{\xi}_\sigma = (\overset{i}{a}_\sigma - \overset{i}{a}_0)/(\Sigma_i(\overset{i}{a}_\sigma - \overset{i}{a}_0)^2)^{1/2} \to \overset{i}{\xi}_0, \\ T^i(\overset{j}{a}_\sigma) = b^i = T^i(\overset{j}{a}_0). \end{gathered} \tag{1}$$

Received by the editors November 5, 1942.

[1] Marston Morse and S. B. Littauer, *A characterization of fields in the calculus of variations*, Proc. Nat. Acad. Sci. U.S.A. vol. 18 (1932) pp. 724–730. This paper will hereafter be designated by M. L.