ON THE RADICAL OF A GENERAL RING

JAKOB LEVITZKI

1. Introduction. The radical of a ring has been hitherto defined by using either the notion of a nilpotent or a nil-ideal. In the following we shall ascribe the term *specialized radical* to the sum N_σ of all two-sided nilpotent ideals of a ring S, and the term *generalized radical* to the sum N_γ of all two-sided nil-ideals of the ring. In §2 of the present note the notions of a *semi-nilpotent* ideal and its counterpart the *semi-regular* ideal are introduced, and the term *radical* is suggested for the sum N of all two-sided semi-nilpotent ideals of the ring. These notations may be justified by the following considerations:

(a) Each nilpotent ideal is semi-nilpotent, and each semi-nilpotent ideal is a nil-ideal.

(b) The radical N is a two-sided semi-nilpotent ideal which contains also all one-sided semi-nilpotent ideals of the ring.

(c) The radical of S/N is zero.

(d) The radical N contains the specialized radical N_σ and is a subset of the generalized radical N_γ.

(e) In the case of an algebra the notions: nilpotent, semi-nilpotent and nil-ideal are identical, and $N_\sigma = N = N_\gamma$; but if one turns to general rings, and replaces radical N and semi-nilpotent ideals either by specialized radical N_σ and nilpotent ideals or generalized radical N_γ and nil-ideals, then some restriction has to be imposed on the ring S in order to assure the validity of (b) and (c).[1]

These results are applied in §3 to semi-primary rings (which will be called in short: A-rings).

2. The radical of a general ring. In this section certain theorems related to the radical of a general ring are proved.

NOTATION. If $r_1, \cdots, r_n$ is a finite set of elements in the ring S, then the ring generated by the r will be denoted by $\{r_1, \cdots, r_n\}$.

DEFINITION. *A right ideal is called semi-nilpotent if each ring gener-*

Presented to the Society, October 31, 1942; received by the editors October 7, 1942.

[1] Thus the specialized radical N_σ is nilpotent (and hence the specialized radical of S/N_σ is zero) if (as is well known) the maximal condition or (as can be shown) the minimal condition is satisfied by the two-sided ideals of the ring. As to the generalized radical $N\gamma$, it has been proved (G. Koethe, *Die Struktur der Ringe*, Math. Zeit. vol. 32 (1930) pp. 161–186) that if each regular right ideal of S contains a minimal regular right ideal, then N_γ contains also all one-sided nil-ideals of the ring.