# ON NON-DENUMERABLE GRAPHS

P. ERDÖS AND S. KAKUTANI

The present paper consists of two parts. In Part 1 we prove a theorem on the decomposition of a complete graph. This result is then applied in Part 2 to show that the continuum hypothesis is equivalent to the possibility of decomposing the set of all real numbers into a countable number of summands each consisting of rationally independent numbers.

## PART 1

A graph $G$ is complete if every pair of points of $G$ is connected by one and only one segment. $G$ is called a tree if it does not contain any closed polygon.

THEOREM 1. *A complete graph of cardinal number $m$ (that is, the cardinal number of the vertices is $m$) can be split up into a countable number of trees if and only if $m \leqq \aleph_1$.*

PROOF. We shall first prove that every complete graph of power $\aleph_1$ can be split up into the countable sum of trees.[1] Let $G$ be a complete graph of cardinal number $\aleph_1$. Let $\{x_\alpha\}$, $\alpha < \omega_1$, be any well ordered set of power $\aleph_1$. We may assume that $G$ is represented by a system of segments $(x_\alpha, x_\beta)$, $\alpha < \beta < \omega_1$. For any $\beta < \omega_1$ arrange the set of all $\alpha < \beta$ into a sequence $\alpha_{\beta,n}$, $n = 1, 2, \cdots$, and let $G_n$ be the set of all segments $(x_\alpha, x_\beta)$ such that $\alpha = \alpha_{\beta,n}$. It is clear that $G = \bigcup_{n=1}^\infty G_n$ and that for each $G_n$, for every $\beta < \omega_1$, there exists one and only one $\alpha$ such that $(x_\alpha, x_\beta) \in G_n$ and $\alpha < \beta$. From this last fact it is clear that $G_n$ does not contain any closed polygon.

Conversely, let us assume that a complete graph $G$ of cardinal number $m$ is split up into a countable number of trees $T_n$; $G = \bigcup_{n=1}^\infty T_n$. We shall prove that $m \leqq \aleph_1$. We can again assume that $G$ is represented by a system of segments $(x_\alpha, x_\beta)$, $\alpha < \beta < \phi$, where $\{x_\alpha\}$, $\alpha < \phi$, is a well ordered set of cardinal number $m$.

We shall first decompose each $T_n$ into four parts $T_{n,i}$, $i = 1, 2, 3, 4$, such that $T_{n,1}$ and $T_{n,2}$ satisfy the condition:

(1) Any two consecutive segments of the graphs $T_{n,1}$ and $T_{n,2}$ are of the form: $(x_\alpha, x_\beta)$, $(x_\alpha, x_\gamma)$, $\alpha < \beta$, $\alpha < \gamma$, $\beta \neq \gamma$. And $T_{n,3}$, $T_{n,4}$ satisfy:

(2) Any two consecutive segments of the graphs are of the form: $(x_\beta, x_\alpha)$, $(x_\gamma, x_\alpha)$, $\beta < \alpha$, $\gamma < \alpha$, $\beta \neq \gamma$.

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[1] This result was also obtained by J. Tukey, oral communication.