THE DISTRIBUTION OF INTEGERS REPRESENTED BY BINARY QUADRATIC FORMS

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R. D. James[1] has proved the following theorem:

THEOREM 1. *Let $B(x)$ denote the number of positive integers $m \leqq x$ which can be represented by positive, primitive, binary quadratic forms of a given negative discriminant d, but are prime to d. Then*

$$B(x) = bx/(\log x)^{1/2} + O(x/\log x), \tag{1}$$

where b is the positive constant given by

$$\pi b^2 = \prod_q (1 - 1/q^2)^{-1} \prod_{p \mid d} (1 - 1/p) \sum_{n=1}^{\infty} (d \mid n) n^{-1}. \tag{2}$$

Here q runs over all primes such that $(d \mid q) = -1$; p denotes any prime greater than or equal to 2; and $(d \mid n)$ is the Kronecker symbol.

We shall deduce from his result an asymptotic formula with the restriction that m be prime to d removed.

First, let p be a prime dividing d but not satisfying

$$p > 2 \text{ and } p^2 \mid d, \quad \text{or} \quad p = 2 \text{ and } d \equiv 0 \text{ or } 4 \pmod{16}. \tag{3}$$

Then pn is represented by p. p. b. q. forms of discriminant d if and only if n is likewise represented.[2] Hence if $p^r \leqq (\log x)^{1/2}$ then the number of represented integers less than or equal to x of the form $p^r m$ with m prime to d, is

$$\frac{bx/p^r}{(\log x/p^r)^{1/2}} + O\left(\frac{x/p^r}{(\log x/p^r)}\right) = \frac{bx/p^r}{(\log x)^{1/2}} + O\left(\frac{x/p^r}{\log x}\right),$$

since $(\log xp^{-r})^{-1/2} - (\log x)^{-1/2} = O((\log p^r)/(\log x)^{3/2})$. Also, if $p^s > (\log x)^{1/2}$,

$$(bx/(\log x)^{1/2})(1/p^s + 1/p^{s+1} + \cdots) = O(x/(\log x)).$$

Hence the number of positive integers less than or equal to x, represented by p.p.b.q. forms of discriminant d, and prime to d except that they need not be prime to p, is given by

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[1] R. D. James, Amer. J. Math. vol. 60 (1938) pp. 737–744.

[2] G. Pall, Math. Zeit. vol. 36 (1933) pp. 321–343, p. 331.