THE PELL EQUATION IN QUADRATIC FIELDS

IVAN NIVEN

Consider the equation

$$\xi^2 - \gamma\eta^2 = 1, \tag{1}$$

where γ is a given integer of a quadratic field F, and integral solutions ξ, η are sought in F. It has been shown[1] that equation (1) has an infinite number of solutions if and only if γ is not totally negative when F is a real field, and γ is not the square of an integer of F when F is imaginary. We now obtain the following result:

Let γ be such that equation (1) *has an infinite number of solutions. If F is a real field it is possible to find a solution ξ_1, η_1 of* (1) *so that every solution is given by the equations*

$$\begin{aligned} \xi &= \{(\xi_1 + \gamma^{1/2}\eta_1)^n + (\xi_1 - \gamma^{1/2}\eta_1)^n\}/2 \\ \eta &= \{(\xi_1 + \gamma^{1/2}\eta_1)^n - (\xi_1 - \gamma^{1/2}\eta_1)^n\}/(2\gamma^{1/2}), \end{aligned} \qquad n = 1, 2, 3, \cdots, \tag{2}$$

if and only if γ is not a totally positive non-square integer of F. If F is imaginary it is always possible to find a solution ξ_1, η_1 so that all solutions are given by (2).

The latter result is known to hold for the Pell equation in the rational field. The expression $\gamma^{1/2}$ is ambiguous, but no confusion will arise provided it consistently has the same value (we shall specify its value in certain cases). We consider the four sets $\pm\xi$, $\pm\eta$ to be a single solution, so that equations (2) give "every solution" in the sense that one of the four is present for some value of n.

Case 1. *F real, γ positive but not totally positive.* It will be convenient to consider $\gamma^{1/2}$, ξ and η positive. We now show that there is but a finite number of solutions of (1) with ξ bounded, say $\xi < N$. For suppose we have an infinitude of solutions ξ_i, η_i with $\xi_i < N$ for $i = 1, 2, 3, \cdots$. Taking conjugates in equation (1) we would have

$$\bar{\xi}_i^2 - \bar{\gamma}\bar{\eta}_i^2 = 1,$$

and since $-\bar{\gamma}$ is positive, this implies that $\bar{\xi}_i \leqq 1$ for $i = 1, 2, 3, \cdots$. But it is not possible to have an infinite set of real quadratic integers which, along with their conjugates, are bounded.

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[1] *Quadratic diophantine equations in the rational and quadratic fields*, Trans. Amer. Math. Soc. vol. 52 (1942) p. 2 Theorem 4. We refer to this paper as (Q).