NEW PROOFS OF THE THEOREMS OF BELTRAMI AND KASNER ON LINEAR FAMILIES

JOHN DECICCO

1. **Introduction.** We submit here new proofs, from a uniform point of view, of the theorems of Beltrami and Kasner on linear families of curves. Beltrami's result is that a surface S may be mapped upon a plane π so that its geodesics correspond to straight lines if and only if S is of constant gaussian curvature [1].[1] Kasner's result states that a complete system of isogonal trajectories of a simple (that is, one-parameter) family of curves F is linear if and only if F is isothermal [2]. We shall also deduce from our work another theorem of Kasner stating that a surface S can possess exactly ∞^2 isothermal families of geodesics (maximum possibility) if and only if S is of constant gaussian curvature [3].

2. **Velocity systems.** For the development of our proofs, it is found necessary to consider certain classes of ∞^2 curves, namely, velocity systems, natural families, isogonal systems, Γ_0 and Γ families. In Kasner's study of dynamical trajectories [4], an important class of ∞^2 curves was encountered which he termed *velocity systems*. In minimal coordinates $(u=x+iy, v=x-iy)$, any such system is defined by a second order differential equation of the form

$$v'' = v'(c - dv'), \tag{1}$$

where c and d are arbitrary functions of (u, v).

Special types of velocity systems are natural families and isogonal systems. Any natural family is a velocity system for which $c_v = d_u$, whereas any isogonal system is a velocity system for which $c_v = -d_u$.

A system of ∞^2 curves is both natural and isogonal if and only if it is the complete set of isogonal trajectories of an isothermal family. Such set is called a *conformal rectilinear wex*[2] and is denoted by Γ_0. Any family of this type is conformally equivalent to the ∞^2 straight lines of the plane. A system Γ_0 is a velocity system for which $c_v = d_u = 0$.

Presented to the Society, October 31, 1942; received by the editors August 5, 1942.

[1] The numbers in brackets refer to the references at the end of the paper.

[2] The set of ∞^2 integral curves of any differential equation of second order $y'' = F(x, y, y')$ has been termed a *wex* by Kasner. The transformation theory of Γ_0 systems has been developed by Kasner and DeCicco, *Transformation theory of isogonal trajectories of isothermal families*, Proc. Nat. Acad. Sci. U.S.A. 1942.