ON LINEAR COMBINATIONS OF QUADRATIC FORMS

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The characteristics of linear combinations $\sum\lambda_i Q_i(x)$ of a given set of real quadratic forms

$$(1) \qquad Q_i(x) \equiv \sum_{k,l=1}^{n} \overset{i}{a}_{kl} x_k x_l, \qquad i = 1, 2, \cdots, m,$$

have been considered in several recent papers.[1]

One of the theorems in my earlier paper may be stated as follows:

A necessary and sufficient condition that there exist a linear combination $\sum\lambda_i Q_i(x)$ which is positive definite is that there exist no set of points $x^j = (x_1^j, x_2^j, \cdots, x_n^j) \neq (0, \cdots, 0)$ $(j = 1, 2, \cdots, r)$ such that

$$(2) \qquad \sum_{j=1}^{r} \mu_j Q_i(x^j) = 0, \qquad i = 1, 2, \cdots, m,$$

the coefficients μ_j being positive.

Shortly after the publication of this paper, Fritz John kindly called my attention to the fact that a closely related result is contained in an earlier paper of his.[2]

Certainly John's paper contains essentially the "sufficiency" half of the theorem quoted above. Furthermore it introduces a very interesting suggestion in noting that the validity of relations (2) implies the existence of a quadratic form

$$B(x) \equiv \sum_{k,l=1}^{n} b_{kl} x_k x_l$$

which is definite or semi-definite, and such that

$$\sum_{k,l=1}^{n} \overset{i}{a}_{kl} \cdot b_{kl} = 0, \qquad i = 1, 2, \cdots, m.$$

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[1] Finsler, *Über das Vorkommen definiter und semidefiniter Formen in Scharen quadratischer Formen*, Comment. Math. Helv. vol. 9 (1937) pp. 188–192. Hestenes and McShane, *A theorem on quadratic forms and its application in the calculus of variations*, Trans. Amer. Math. Soc. vol. 47 (1940) pp. 501–512. Dines, *On the mapping of n quadratic forms*, Bull. Amer. Math. Soc. vol. 48 (1942) pp. 467–471.

[2] *A note on the maximum principle for elliptic differential equations*, Bull. Amer. Math. Soc. vol. 44 (1938) pp. 268–271.