SECTIONS OF CONTINUOUS COLLECTIONS

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In the present note we establish the following

THEOREM. *Suppose G is a continuous collection*[1] *of closed and compact sets filling a separable metric space X. Suppose further that the space G, considered as a decomposition space, has dimension at most n. Then there is a closed subset K of X, such that for each $g \in G$, the set $g \cdot K$ is nonvacuous and consists of at most $(n+1)$ points.*

We call such a point set K an $(n+1)$-section of the collection G. Thus a 1-section of G is a true section. G. T. Whyburn[2] has shown that if the elements of G are 0-dimensional and G is a dendrite, then G admits a true section. The present result gives only a 2-section, but there is no hypothesis on the dimension of the elements of G. For $n=1$, it is known that in general G does not admit a true section. For $n>1$ it is not known whether the present result gives the best possible constant.

We first establish the theorem in the 0-dimensional case.

LEMMA. *Suppose G is 0-dimensional, and ϵ is a given positive number. Suppose W is an open set in X such that $W \cdot g \neq 0$ for each $g \in G$. Then there is an open set E in X such that $\overline{E} \subset W$, $E \cdot g \neq 0$ for every $g \in G$, and the diameter of $E \cdot g < \epsilon$ for each $g \in G$.*

Let $f(x)$ be a homeomorphism of M, a subset of the Cantor set, into G.[3] In the product space $M \times X$, consider the set A of points (x, y) with $x \in M$ and $y \in f(x)$. For $x \in X$ there is a unique $y = y(x)$ in M such that $x \in f(y)$. The function $t(x) = (y(x), x)$ is a homeomorphism of X into A.

In the space A, the open set $t(x)$ and the continuous collection H of elements $t(g)$ for $g \in G$ satisfy the properties of W and G stated in the hypothesis of the lemma. Furthermore, the diameter of a set Z in A is not smaller than the diameter of $t^{-1}(Z)$. Hence all we need show is that there exists an open set E satisfying the theorem relative to the open set $t(W) = U$ and the continuous collection H.

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[1] *A continuous collection filling a space X*, is a collection G of sets g such that: (1) If $x \in X$, then $x \in g$ for exactly one g. (2) If $x \in g$, $x_n \in g_n$ and $x_n \to x$, then $\lim g_n = g$.

[2] *A theorem on interior transformations*, Bull. Amer. Math. Soc. vol. 44 (1938) pp. 414–416.

[3] P. Urysohn, *Sur les multiplicités Cantoriennes*, Fund. Math. vol. 7 (1926) p. 77.