ON THE JOIN OF TWO COMPLEXES

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1. **Introduction.** In this note we point out an isomorphism between the $(r+1)$-dimensional Betti group of the join (defined below) of two complexes and a subgroup of the r-dimensional Betti group of the product of the two complexes. Using this isomorphism the Betti groups of the join are derived from those of the product in case the complexes are finite.[1]

2. **Definition of the join (K_1, K_2) of K_1 and K_2.** To define the join of two complexes we first define the join (σ, τ) of a p-dimensional simplex σ and a q-dimensional simplex τ, $p, q=0, 1, \cdots$. This join is a $(p+q+1)$-dimensional simplex with a p-dimensional side associated with σ and the opposite side, which is q-dimensional, associated with τ. These sides will not be distinguished from σ and τ, respectively. Now consider the complexes K_1 and K_2. Consider the set consisting of the simplexes σ_α of K_1, the simplexes τ_β of K_2, and the simplexes $(\sigma_\alpha, \tau_\beta)$. In a natural way this set forms a complex. We define the *join* (K_1, K_2) of K_1 and K_2 to be the first barycentric subdivision of this complex.

3. **The rays.** By the rays of (σ, τ) we mean the straight line segments each of which joins a point of σ and a point of τ. These rays cover (σ, τ). Also no two rays intersect except possibly at an end point. The rays of all $(\sigma_\alpha, \tau_\beta)$ of (K_1, K_2) are called the *rays*.

Let N_i, $i=1, 2$, be the subcomplex made up of the simplexes of (K_1, K_2) that have at least one vertex in K_i together with the faces of all such simplexes. It is known that each ray meets the intersection $N_1 \cap N_2$ in exactly one point.[2] Furthermore N_i and $N_1 \cap N_2$ can be homotopically deformed in N_i along the rays into K_i, $i=1, 2$.[2] It follows that $N_1 \cap N_2$ and the product $K_1 \times K_2$ are homeomorphic (the complexes being considered as point sets).

4. **The theorem.** We prove this theorem.

THEOREM 1. *There is an isomorphism between the $(r+1)$-dimensional*

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[1] The Betti groups of the join of two finite complexes are known. They were computed by H. Freudenthal in his paper *Die Bettischen Gruppen der Verbindung Zweier Polytope*, Fund. Math. vol. 29 (1937) pp. 145–150.

[2] For a proof see our paper *Simultaneous invariants of a complex and subcomplex*, Duke Math. J. vol. 5 (1939) pp. 62–71.