A FAMILY OF FUNCTIONS AND ITS THEORY OF CONTACT[1]

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Introduction. If $p_1, \cdots, p_n$ are fixed positive integers and $a_1, \cdots, a_n$ arbitrary constants, it is possible so to choose the a_i as to make the function

$$(1) \qquad y(x) = \prod_{i=1}^{n} (x - a_i)^{p_i}$$

and its first $p_1 + \cdots + p_n - 1$ derivatives equal to zero for any single value x_0 of x. This is accomplished by taking each a_i equal to x_0. One might say, on this basis, that *the family of polynomials* (1) *has contact of order* $p_1 + \cdots + p_n - 1$, *for every value of* x, *with* $y = 0$.

A more interesting situation is met when we allow the p_i to be any fixed positive numbers, not necessarily integral. In that case $y(x)$ may be a function of many branches, with the quotient of any two branches equal to a constant of modulus unity. For our purposes it suffices to consider the value zero of x. If no a_i is zero, each branch of $y(x)$ will be analytic at $x = 0$, with an expansion

$$c_0 + c_1 x + \cdots + c_s x^s + \cdots$$

where the c_j depend on the a_i. The question which we examine is: *What is the greatest value of s such that, by suitably varying the a_i, the coefficients $c_0, \cdots, c_s$ can be made to approach zero simultaneously?* Such a greatest value of s exists, and will be called, below, *the order of contact of the family* (1) *with* $y = 0$. Denoting the greatest value of s by r, we shall prove that

$$(2) \qquad r \leqq q + n - 1$$

where q is the greatest integer less than $p_1 + \cdots + p_n$. When no proper subset of the p_i has an integral sum, the equality sign holds in (2). For $n = 2$, (2) can be an inequality only when p_1 and p_2 are both integers. For $n \geqq 3$, (2) will certainly be an inequality if some integral power of $y(x)$ is a polynomial of degree not exceeding $q + n - 1$; thus the order of contact of the family

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[1] The problem of this note was suggested by the considerations of our paper *On the singular solutions of algebraic differential equations*, Ann. of Math. (2) vol. 37 (1936) p. 552. See also, W. C. Strodt, Trans. Amer. Math. Soc. vol. 45 (1939) p. 276.