ON THE CONVERGENCE OF SEQUENCES OF LINEAR OPERATIONS

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1. Introduction. Let S be a sequence $\{U_n(x)\}(n=1, 2, \cdots)$ of linear operations defined over the elements $\{x\}$ of a Banach space[1] E with values lying in a second B space E_1. The fundamental convergence theorem associated with such sequences may be stated as follows:

BANACH-STEINHAUS THEOREM. *For the convergence of S over E it is necessary and sufficient that S converge over a set H dense in a sphere K of E and that the set of norms $\{|U_n|\}$ be bounded.*

The solution of convergence problems involving sequences of positive functionals defined on certain function-spaces[2] led the author to investigate a different form of this convergence criterion which seems to have a more direct practical application in some special cases of importance. The development hinges on the fact that from a certain point of view every linear operation has domains over which it is "positive."

2. A new convergence criterion. Let us make the following definition:

DEFINITION. *A set of points P in a normed vector space will be called a p-set provided that, for every pair of points z and z' of P, $\|z-z'\| \leqq \|z+z'\|$.*

Such a set has many of the characteristic properties of a set of real numbers all of the same sign. In effect, we have extended the notion of "positiveness" to the elements of an abstract space.

In terms of this definition we may now state the following theorem:

THEOREM 1. *For the convergence of S over E, it is sufficient that there exist a sphere K of E such that* (a) *S converges over a set H dense in K and* (b) *for all fixed n sufficiently large, the set of transforms $\{U_n(x)\}$ $(x \subset K)$, is a p-set, P_n.*

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[1] A space of type B. Cf. Banach, *Théorie des opérations linéaires*, Warsaw, 1932, p. 53. In general we follow the terminology of this treatise.

[2] R. P. Bailey, *Convergence of sequences of positive linear functional operations*, Duke Math. J. vol. 2 (1936) pp. 287–303.