THE MODULAR SPACE DETERMINED BY A POSITIVE FUNCTION

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At the suggestion of T. H. Hildebrandt the authors undertook to determine the nature of the space of modular functions of E. H. Moore when the range $\mathfrak{P}$ is taken to be the infinite interval $-\infty < x < +\infty$ and the base matrix ϵ to be of the form

$$(1) \qquad \epsilon(x, y) = \int_{-\infty}^{+\infty} e^{i(x-y)t} dV(t),$$

where V is a monotonically increasing bounded function. This form of ϵ is suggested by the work of Bochner on positive functions.[1] In this note we determine the form of functions modular as to ϵ and of the J-integral.

To avoid, at first, convergence questions we turn our attention to functions ϕ finite as to ϵ, that is, functions of the form

$$(2) \qquad \phi(x) = \sum_{j=1}^{n} \epsilon(x, y_j)a_j = \int_{-\infty}^{+\infty} e^{ixt}\lambda(t)dV(t),$$

where

$$(3) \qquad \lambda(t) = \sum_{j=1}^{n} a_j e^{-iy_j t}.$$

In the formulas (2) and (3) the a_j are arbitrary constants and the y_j are points on the interval $(-\infty, +\infty)$. It is known from standard results in the theory of modular and finite functions[2] that every function ϕ finite as to ϵ is modular and that

$$(4) \qquad \begin{aligned} N\phi &= J\bar{\phi}\phi = \sum_{j,k=1}^{n} \bar{a}_j \epsilon(x_j, x_k) a_k, \\ J\bar{\phi}_1\phi_2 &= [N(\phi_1 + \phi_2) - N(\phi_1 - \phi_2) - iN(\phi_1 + i\phi_2) \\ &\qquad + iN(\phi_1 - i\phi_2)]/4. \end{aligned}$$

Calculating the values of $N\phi$ and $J\bar{\phi}_1\phi_2$, we see that

$$J\bar{\phi}_1\phi_2 = \int_{-\infty}^{+\infty} \bar{\lambda}_1\lambda_2 dV, \qquad N\phi = \int_{-\infty}^{+\infty} |\lambda|^2 dV.$$

Received by the editors March 7, 1942.

[1] S. Bochner, *Monotone Funktionen, Stieltjessche Integrale und harmonische Analyse*, Mathematische Annalen, vol. 108 (1933), pp. 378–410.

[2] E. H. Moore, *General Analysis*, Part II, Philadelphia, 1939, pp. 94 ff.