HOMOGENEOUS AND NONHOMOGENEOUS DIOPHANTINE EQUATIONS

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In an earlier paper[1] we considered homogeneous polynomials f and g whose degrees are relatively prime, and solved the Diophantine equation $f(x)=g(y)$. These results are generalized in the present paper. The solutions are given in terms of arbitrary parameters and if the parameters are integral the solutions are also.

We begin our discussion with the hypothesis that the functions $f(x)=f(x_{11}, \cdots, x_{1m}, \cdots, x_{n1}, \cdots, x_{nm})$, $g(x)=g(x_{11}, \cdots, x_{1m}, \cdots, x_{n1}, \cdots, x_{nm})$ are polynomials with integral coefficients, homogeneous in each set of variables $x_{k1}x_{k2} \cdots x_{km}$; f being of degree $\alpha_k \geqq 0$, g being of degree $\beta_k \geqq 0$ in the sets $x_{k1}x_{k2} \cdots x_{km}$ and $d_k=\alpha_k-\beta_k$. We suppose further that integers λ_i, $\mu_i \geqq 0$ exist such that[2]

$$\sum_{i=1}^{n} d_i\lambda_i = -\sum_{i=1}^{n} d_i\mu_i = 1. \tag{1}$$

THEOREM 1. *The Diophantine equation*

$$f(x) = g(x) \tag{2}$$

has integral solutions, and every solution for which the members of (2) *do not vanish, is equivalent* (*in a sense to be defined*) *to one of the solutions given by*

$$x_{kj} = \alpha_{kj}[g(\alpha)]^{\lambda_k}[f(\alpha)]^{\mu_k}, \qquad k = 1, \cdots, n;\ j = 1, \cdots, m, \tag{3}$$

where the α_{kj} *are arbitrary integers.*

PROOF. Let $x_{kj}=\alpha_{kj}s^{\lambda_k}t^{\mu_k}$. Then by (1), (2) becomes[3] $sf(\alpha)=tg(\alpha)$, which is satisfied identically in the α_{kj} if $s=g(\alpha)$, $t=f(\alpha)$. Hence (3) is a solution of (2).

We now define the concept of equivalent solutions. Suppose $x_{kj}=\rho_{kj}$ is a solution of (2). If there are no integers $b>1$, ρ'_{kj} such that $\rho_{kj}=\rho_{kj}b^{\sigma_k}$, where the σ_k are positive integers such that $\sum_{k=1}^{n}\alpha_k\sigma_k$

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[1] A. A. Aucoin and W. V. Parker, *Diophantine equations whose members are homogeneous*, this Bulletin, vol. 45 (1939), pp. 330–333.

[2] We need to postulate only the existence of λ_i since it may be shown that for n odd $\mu_i=\sum_{j=1}^{n}d_j-d_i-2\sum_{k=1}^{(n-1)/2}d_{i+k}-\lambda_i$ where $d_{n+p}=d_p$, and for n even $\mu_i=\sum_{j=1}^{n}d_j-d_i-2\sum_{k=1}^{(n/2)-1}d_{i+k}-2d'_{n/2+i}-\lambda_i$ where $d'_k=d_k$ for $k\leqq n$, $d'_k=0$ for $k>n$ and $d_{n+p}=d_p$. We must, however, assume that $\mu_i\geqq 0$.

[3] It will be shown later that s, $t\neq 0$.