ON VALUE REGIONS OF CONTINUED FRACTIONS

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The results of this paper are contained in the following theorem.

THEOREM 1. *If the elements* $a_1, a_2, \cdots$ *of the continued fraction*

$$1+\frac{a_1}{1}+\frac{a_2}{1}+\cdots \tag{1}$$

lie in the parabola

$$\rho \leqq \frac{2d(1-d)}{1-\cos\theta}, \qquad 1/2 \leqq d < 1, \tag{2}$$

then the approximants of the continued fraction (1) *lie in the hyperbolic region*

$$R > \frac{2d(1-d)}{1-2d+\cos\phi}, \qquad -\beta < \phi < \beta, \tag{3}$$

where $\beta = \text{arc cos } (2d-1)$. *If* z *is any value on the boundary of the region* (3), *there exists one and only one continued fraction of the form* (1), *with elements in the parabola* (2), *converging to* z, *namely:*[1]

$$1+\frac{a}{1}+\frac{\bar{a}}{1}+\frac{a}{1}+\cdots \tag{4}$$

where $a=(z-1)\bar{z}$ *is a value on the boundary of the parabola* (2).

For the case $d=1/2$, Scott and Wall [3] determined the value region of the approximants and Paydon [1] established the uniqueness property of (4) for that case.

A convergence criterion due to Scott and Wall [2] insures the convergence of the continued fraction (1) if in addition to the conditions of Theorem 1 it is required that the series $\sum |b_n|$ diverges, where $b_1=1/a_1$, $b_{n+1}=1/a_{n+1}b_n$. The value of such a continued fraction must lie in the closure of the region (3). Finally it follows from Theorem 1 that all values in the region (3) are assumed by a continued fraction of the form (4). The following result is now seen to be a consequence of Theorem 1.

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[1] Here and elsewhere in the paper a bar over a number means the complex conjugate of the number.