THE RADICAL OF A NON-ASSOCIATIVE ALGEBRA

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1. **Introduction.** An algebra $\mathfrak{A}$ is said to be nilpotent of index r if every product of r quantities of $\mathfrak{A}$ is zero, and is said to be a zero algebra if it is nilpotent of index two. It is said to be simple if it is not a zero algebra and its only nonzero (two-sided) ideal is itself, and is said to be semi-simple if it is a direct sum of simple algebras.

The radical of an associative algebra $\mathfrak{A}$ is a nilpotent ideal $\mathfrak{N}$ of $\mathfrak{A}$ which is maximal in the strong sense in that it contains[1] all nilpotent ideals of $\mathfrak{A}$. No such ideal exists in an arbitrary non-associative algebra, and so the radical of such an algebra has never[2] been defined. However the property that $\mathfrak{A}-\mathfrak{N}$ be semi-simple is really the vital one and we shall define the concept of radical here by proving this theorem.

THEOREM 1. *Every algebra $\mathfrak{A}$ which is homomorphic to a semi-simple algebra has an ideal $\mathfrak{N}$, which we shall call the* **radical** *of $\mathfrak{A}$, such that $\mathfrak{A}-\mathfrak{N}$ is semi-simple, $\mathfrak{N}$ is contained in every ideal $\mathfrak{B}$ of $\mathfrak{A}$ for which $\mathfrak{A}-\mathfrak{B}$ is semi-simple.*

The hypothesis that $\mathfrak{A}$ shall be homomorphic to a semi-simple algebra is equivalent to the property that there shall exist an ideal $\mathfrak{B}$ in $\mathfrak{A}$ such that $\mathfrak{A}-\mathfrak{B}$ shall be semi-simple. It is a necessary assumption even in the associative case, since $\mathfrak{A}$ may be nilpotent and then $\mathfrak{A}=\mathfrak{N}$, every $\mathfrak{A}-\mathfrak{B}$ is nilpotent. Moreover it is satisfied by every algebra $\mathfrak{A}$ with a unity quantity. We shall, nevertheless, carry our study a step farther in that we shall define explicitly a certain proper ideal $\mathfrak{N}$ for every algebra $\mathfrak{A}$ such that either $\mathfrak{N}$ is the radical of $\mathfrak{A}$ in the sense above or $\mathfrak{A}$ is not homomorphic to a semi-simple algebra. In the latter case $\mathfrak{A}-\mathfrak{N}$ is a zero algebra.

Our results will be consequences of the remarkable fact[3] that the major structural properties of any non-associative algebra $\mathfrak{A}$ over $\mathfrak{F}$ are determined by almost the same properties of a certain related associative algebra $T(\mathfrak{A})$. We define the right multiplications R_x and the

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[1] For these results see my *Structure of Algebras*, American Mathematical Society Colloquium Publications, vol. 24, 1939, chap. 2.

[2] For the case of alternative algebras see M. Zorn, *Alternative rings and related questions* I: *Existence of the radical*, Annals of Mathematics, (2), vol. 42 (1941), pp. 676–686.

[3] Cf. my *Non-associative algebras* I: *Fundamental concepts and isotopy*, Annals of Mathematics, (2), vol. 43 (1942), pp. 685–708. See also N. Jacobson, *A note on non-associative algebras*, Duke Mathematical Journal, vol. 3 (1937), pp. 544–548.