THE MEASURE OF THE CRITICAL VALUES OF DIFFERENTIABLE MAPS

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1. Introduction. Consider the *map*

$$y^j = f^j(x^1, x^2, \cdots, x^m), \qquad j = 1, 2, \cdots, n, \tag{1.1}$$

of a region R of euclidean m-space into part of euclidean n-space. Suppose that each function f^j $(j=1, \cdots, n)$ is of class C^q in R $(q \geqq 1)$.[1] A *critical point* of the map (1.1) is a point in R at which the matrix of first derivatives $\mathfrak{M} = \|f_i^j\|$ $(i=1, \cdots, m; j=1, \cdots, n)$ is of less than maximum rank. The *rank* of a critical point x is the rank of $\mathfrak{M}$ at x. A *critical value* is the image under (1.1) of a critical point. If $n=1$, these definitions are the usual definitions of critical point and value of a continuously differentiable function.

We prove the following result: *If $m \leqq n$, the set of critical values of the map* (1.1) *is of m-dimensional measure*[2] *zero* without further hypothesis on q; *if $m > n$, the set of critical values of the map* (1.1) *is of n-dimensional measure zero providing that $q \geqq m-n+1$.* Using an example due to Hassler Whitney[3] we show that the hypothesis on q cannot be weakened. We prove also that the critical values of (1.1) corresponding to critical points of rank zero constitute a set of (m/q)-dimensional measure zero.

The idea of considering the measure of the set of critical values of one function or of several functions is due to Marston Morse.

The first result stated above reduces, if $n=1$, to the known theorem: The critical values of a function of m variables of class C^m constitute a set of linear measure zero. A. P. Morse[4] has given a proof of this theorem for all m. In the present paper we make use of one of A. P. Morse's results.

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[1] A function is of class C^q if all its partial derivatives of order q exist and are continuous.

[2] In the sense of Hausdorff-Saks. The definition is given in §2.

[3] H. Whitney, *A function not constant on a connected set of critical points*, Duke Mathematical Journal, vol. 1 (1935), pp. 514–517.

[4] A. P. Morse, *The behaviour of a function on its critical set*, Annals of Mathematics, (2), vol. 40 (1939), pp. 62–70. Proofs for the cases $m=1, 2, 3$ had previously been given by M. Morse and for the cases $m=4, 5, 6$ by M. Morse and the author in unpublished papers.