SYMMETRIC DIFFERENTIAL EXPRESSIONS

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1. Introduction. In a recent article,[1] it is shown that if

$$f(x_1, \cdots, x_n) = \sum c_{\nu_{01},\cdots,\nu_{kn}} x_1^{\nu_{10}} \cdots (x_1^{(1)})^{\nu_{11}} \cdots (x_n^{(k)})^{\nu_{kn}}, \tag{1}$$

where $x_i^{(k)}$ represents the kth derivative of x_i and $f(x_1, \cdots, x_n)$ is unchanged by all permutations of the variables, then

$$f(x_1, \cdots, x_n) = \frac{1}{D^p} \sum C_{\lambda_{01},\cdots,\lambda_{kn}} a_1^{\lambda_{10}} \cdots (a_1^{(1)})^{\lambda_{11}} \cdots (a_n^{(k)})^{\lambda_{kn}} \tag{2}$$

where D is the discriminant of $x_1, \cdots, x_n$, and $a_1, \cdots, a_n$ are the elementary symmetric functions or E.S.F.'s. In applications to problems involving the differential equations satisfied by algebraic functions or the algebraic properties of the solutions of algebraic differential equations, it is desirable to have some method for passing from (1) to (2). The proof of the basic theorem, although constructive, gives a method that is prohibitively laborious and will often introduce unnecessary powers of D. It is the object of this paper to present an exhaustion procedure which greatly simplifies the work and obviates the latter danger.

2. Preliminaries. It is no restriction to limit ourselves to symmetric differential functions generated by a single term of (1). Term A of such a function will be said to be of higher *order* than term B if the first exponent ν_{ij} in A which differs from the corresponding exponent ν'_{ij} in B is the larger.

THEOREM 1. *The exponents of any power product of the derivatives of the E.S.F.'s are uniquely determined by the highest order term; and the exponents ν_{ij} of the highest order term satisfy the inequalities, where $j=1, \cdots, n-1$,*

$$\nu_{0j} - \nu_{0j+1} \geqq \sum_{i=1}^{k} \nu_{ij+1}. \tag{3}$$

PROOF. It is clear that the highest order term in $a_i^{(j)}$ is $x_1 \cdots x_{i-1} x_i^{(j)}$ and that the highest order term in

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[1] H. J. Riblet, *Algebraic differential fields*, American Journal of Mathematics, vol. 63 (1941), p. 341.