# A GENERALIZATION OF THE POLAR REPRESENTATION OF NONSINGULAR MATRICES

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1. **Introduction.** If $A$ is a square matrix with elements in the complex number field, then

$$(1) \qquad A = PU,$$

where $P$ is a positive definite hermitian matrix and $U$ is a unitary matrix.[1] In this polar representation of the matrix $A$, as it is called, the two matrices $P$ and $U$ are unique. Since the matrix $P$ is positive definite and nonsingular, it has the same signature as the identity matrix $E$ while the unitary matrix $U$ is a conjunctive automorph of $E$. From (1) we may deduce a somewhat similar representation of $A$ in terms of a positive definite hermitian matrix and a conjunctive automorph, not of $E$, but of any nonsingular positive definite hermitian matrix.

Let $H$ be a nonsingular hermitian matrix which is positive definite, so that there exists a nonsingular matrix $Q$ satisfying

$$Q^{-1}H(Q^{-1})^* = E,$$

where $Q^*$ is the conjugate transposed of $Q$. If $B=Q^{-1}AQ$ and $B=PU$ is the polar representation of $B$, then $A=QPQ^{-1}QUQ^{-1}=DR$, where $D=QPQ^{-1}$ and $R=QUQ^{-1}$. Since $DH=QPQ^{-1}QQ^*=QPQ^*$, $DH$ is hermitian with the same signature as $H$. Further $RHR^*=QUQ^{-1}QQ^* \cdot(Q^{-1})^*U^*Q^*=QQ^*=H$. Hence we have this result as an analogue of the polar representation (1) of $A$.

RESULT (1). *If $H$ is any nonsingular positive definite hermitian matrix and $A$ is a nonsingular matrix, then*

$$(2) \qquad A = DR,$$

*where $DH$ is a positive definite hermitian matrix and $RHR^*=H$.*

If $H=H^{-1}$, $A=DR=DHHR$ and, since $HRH(HR)^*=H^3=H$, $A=P_1R_1$, where $P_1$ is a positive definite hermitian matrix and $R_1HR_1^*=H$. Therefore we have as a second analogue of (1) the following result.

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[1] L. Autonne, *Sur les groupes linéaires, réels et orthogonaux*, Bulletin de la Société Mathématique de France, vol. 30 (1902), pp. 121–134. A. Wintner and F. D. Murnaghan, *On a polar representation of non-singular matrices*, Proceedings of the National Academy of Sciences, vol. 17 (1931), pp. 676–678.