HAUSDORFF MEANS INCLUDED BETWEEN $(C, 0)$ AND $(C, 1)$

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In this paper we show that if $\phi(u)$ is any function of bounded variation on the interval $0 \leqq u \leqq \infty$ and $\phi(\infty)-\phi(0)=1$, then the function $\alpha(z)=\int_0^\infty d\phi(u)/(1+zu)$ is a regular moment function; and we show that when $\phi(u)$ is further restricted to be monotone then the Hausdorff mean determined by $\alpha(z)$ is included between $(C, 0)$ and $(C, 1)$. Conditions under which this mean is equivalent to $(C, 0)$ or to $(C, 1)$ are obtained which are analogous to the conditions found by Scott and Wall[1] for the special case where $\phi(u) \equiv 1$ for $u \geqq 1$, $\phi(0)=0$. In §1 we give an elementary development of the notion of Hausdorff summability; §2 contains a proof that $\alpha(z)$ is a regular moment function; §3 contains the above mentioned inclusion theorems; and §4 contains examples and a discussion of some transformations of moment functions which are suggested by the earlier developments.

1. **Hausdorff summability.** Let $A=(a_{ij})$ be any matrix in which $a_{ii} \neq 0$ and $a_{ij}=0$ for $j>i$, $i, j=0, 1, 2, \cdots$, and consider the system of equations

$$
\begin{aligned}
&a_{00}q_0 &&= c_0(a_{00}p_0),\\
&a_{10}q_0 + a_{11}q_1 &&= c_1(a_{10}p_0 + a_{11}p_1),\\
&a_{20}q_0 + a_{21}q_1 + a_{22}q_2 &&= c_2(a_{20}p_0 + a_{21}p_1 + a_{22}p_2),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots .
\end{aligned}
\tag{1.1}
$$

These equations constitute a linear transformation of the sequence $\{p_n\}$ into the sequence $\{q_n\}$, the transformation depending upon the matrix A and the sequence $\{c_n\}$. If $\lim q_n=p$, we shall say that the sequence $\{p_n\}$ is $[A, c_n]$-*summable* to the limit p. A sequence $\{c_n\}$ such that $[A, c_n]$ sums every convergent sequence to its proper limit will be called A-*regular*. The following statements are almost obvious consequences of the above definitions:

(i) If $[A, c_n]$ transforms $\{p_n\}$ into $\{q_n\}$, and $[A, d_n]$ transforms $\{q_n\}$ into $\{r_n\}$, then $[A, c_nd_n]$ transforms $\{p_n\}$ into $\{r_n\}$.

(ii) If $\{c_n\}$, $\{d_n\}$ are A-regular, then $[c_nd_n]$ is A-regular.

(iii) If $[A, c_n]$ sums $\{p_n\}$ to the limit p, then $[A, kc_n]$ sums $\{p_n\}$ to the limit kp.

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[1] W. T. Scott and H. S. Wall, *Transformation of series and sequences*, Transactions of this Society, vol. 51 (1942), pp. 255–279.