PERRON INTEGRALS

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At the recent conference on the theory of integration held at the University of Chicago Professor E. J. McShane remarked that formulas for integration by parts and for change of variable in the integrand have been established for the special Denjoy integral, and consequently for the Perron integral, since the two integrals are equivalent. He then raised the question as to the possibility of proving these formulas directly from the Perron definitions. Later, as we were waiting at the station for my train to leave, Professor Graves talked over with me the implications in McShane's point of view. The following solutions are the outcome of this conversation.

Integration by parts. *Let $g(x)$ be a continuous function on the interval (a, b) for which g' is finite except possibly for a denumerable set, and summable. Let the function $f(x)$ be integrable in the Perron sense and set*

$$F(x) = \int_a^x f(x)dx.$$

Then $f(x)g(x)$ is integrable in the Perron sense, and

$$\int_a^b f(x)g(x)dx = \left[F(x)g(x)\right]_a^b - \int_a^b F(x)g'(x)dx.$$

The conditions on g' make g absolutely continuous, and consequently the difference of two non-decreasing functions each of which is positive or zero. Hence we need consider only the case in which $g(x)$ is non-decreasing and positive or zero. Let $\psi(x)$ be a major function to $f(x)$. Consider $D_*(\psi g)$ which is the lower limit as $h \to 0$ of

$$\frac{\psi(x+h)g(x+h) - \psi(x)g(x)}{h}$$
$$= \frac{g(x+h)\{\psi(x+h) - \psi(x)\}}{h} + \frac{\psi(x)\{g(x+h) - g(x)\}}{h}.$$

From the continuity of g and the fact that both g and g' are not less than zero it follows that

$$D_*(\psi g) \geqq gD_*\psi + \psi g'$$

at every point for which g' exists. Then, since $D_*\psi > -\infty$ except for a

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