ON THE THEORY OF THE TETRAHEDRON

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I. DEFINITION. *We associate with the general tetrahedron* $(T) = ABCD$ *a sphere* (Q) *whose center is the Monge point* M *of* (T) *and the square of whose radius is*

$$\text{(a)} \qquad q^2 = (MO^2 - R^2)/3,$$

where O *and* R *are the center and the radius of the circumsphere* (O) *of* (T).

In what follows, a number of propositions regarding the sphere (Q) will be established and it will be shown that from the properties of (Q) may be derived, as special cases, properties of the polar sphere (H) of the orthocentric tetrahedron (T_h).

For want of a better name we shall refer to (Q) as the "quasi-polar" sphere of the general tetrahedron (T).

The expression $MO^2 - R^2$ is the power of the Monge point M of (T) for the sphere (O).

THEOREM 1. *The square of the radius of the quasi-polar sphere of the general tetrahedron is equal to one-third of the power of the Monge point of the tetrahedron for its circumsphere.*

The sphere (Q) is real, a point sphere, or imaginary according as MO is greater than, equal to, or smaller than R. Moreover, we have $MO < 2R$, for the mid-point of MO is the centroid G of (T), and G necessarily lies within the sphere (O).

COROLLARY. *In an orthocentric tetrahedron* (T_h) *the Monge point coincides with the orthocenter* H, *and the above properties of* (Q) *are valid for the polar sphere* (H) *of* (T_h).[1]

The Monge point M of (T) is a center of similitude of the circumsphere (O) and the twelve point sphere (L) of (T),[2] hence M is the center of a sphere of antisimilitude of (O) and (L), that is, a sphere with respect to which the spheres (O) and (L) are inverse of one another.

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[1] Nathan Altshiller-Court, *Modern Pure Solid Geometry*, New York, 1935, p. 265, §813. This book will be referred to as MPSG.

[2] MPSG, p. 251, §764.