A SUFFICIENT CONDITION FOR CESÀRO SUMMABILITY

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S. Chapman[1] has proved the summability (C, k) of the series $\sum r^s e^{r i\theta}$ where $(k-1) \leqq s < k$, and $0 < \theta < 2\pi$. More recently M. S. MacPhail[2] has proved the exact summability (C, k) of the series $\sum P(r)\cdot f(r)$, where $P(r)$ is a polynomial of degree $k-1$ and $f(r)$ is a periodic function of mean value zero. He also gave a closed expression for the Cesàro "sum" of such a series. H. L. Garabedian[3] earlier published a special case of this result.

The purpose of the present paper is to prove the following closely related theorem in which the condition on the coefficients is somewhat more general. Also "sum" formulas are given for each of the sine and cosine series separately.

THEOREM. *The series* $(1/2)P(0)+\sum P(r)\cos rx$ *and* $\sum P(r)\sin rx$ *are summable* (C, k) *provided* $\{\Delta^k P(r)\}$ *is a monotone null sequence, and* $P(r)$ *together with its first* $k+1$ *derivatives each exist and are continuous for positive values of* r $(x \neq 2n\pi$ *in the cosine series*).

The proofs for the sine and cosine series are almost identical so only the proof for the latter will be given. Consider the series

$$S_n = (1/2)P(0) + \sum_{r=1}^{n} P(r)\cos rx. \tag{1}$$

If we multiply both sides of (1) successively k times by $2\sin x/2$ there results, for k even,

$$\begin{aligned}(2^k \sin^k x/2)S_n &= (1/2)D_{0k} + \sum_{r=1}^{(k-2)/2} D_{rk}\cos rx \\ &\quad + (-1)^{k/2}\sum_{r=0}^{n-k}\Delta^k P(r)\cos(r+(k/2))\,x + (\mathrm{Cos})_k,\end{aligned} \tag{2}$$

where the symbol $(\mathrm{Cos})_k$ represents k cosine terms, the highest order of any of the coefficients being the same as the order of $P(n)$.

For k odd, we have

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[1] S. Chapman, Proceedings of the London Mathematical Society, (2), vol. 9 (1911), p. 398.

[2] M. S. MacPhail, this Bulletin, vol. 47 (1941), p. 483.

[3] H. L. Garabedian, this Bulletin, vol. 45 (1939), p. 592.