GENERALIZATIONS OF THE BERNOULLI POLYNOMIALS AND NUMBERS AND CORRESPONDING SUMMATION FORMULAS

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The Bernoulli polynomials and numbers have been generalized by Nörlund[1] to the Bernoulli polynomials and numbers of higher order. The Bernoulli numbers have been generalized by Vandiver. Analogous polynomials and sets of numbers have been defined from time to time, witness the Euler polynomials and numbers and the so-called Bernoulli polynomials of the second kind.[2]

In the present paper a generalization is made which includes all the above and many other interesting classes of polynomials and corresponding sets of numbers. As a matter of fact the definition of new classes of polynomials by the processes of this paper is a simple matter. In this connection particular attention is called to 2, (d), (h), (i), (j), (k), (l).

An important part of the paper is the development of a whole category of summation formulas related to the studied polynomials as the classical Euler-Maclaurin[3] formula is related to the Bernoulli polynomials or as Taylor's formula is related to $(x-a)^n$.

It will be observed that the work could be varied in detail resulting in closely related polynomials and numbers to those which are obtained. The particular procedure adopted is chosen so as to generalize the Bernoulli polynomials and numbers as now usually defined.[4]

1. **Definition of the polynomials and numbers.** Let us be given two linear operators P and Q with their inverses P^{-1} and Q^{-1}. We shall assume that P reduces the degree of any polynomial by 1 and that Q reduces the degree of any polynomial by $k \geqq 0$, that P operating on a constant gives zero and that Q operating on any polynomial of lesser degree than k gives zero. We assume that P, P^{-1}, Q, Q^{-1} each, where

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[1] N. E. Nörlund, *Differenzenrechnung*, p. 119. For other generalizations see H. S. Vandiver, Proceedings of the National Academy of Sciences, vol. 23, p. 555. See also Leonard Carlitz, this Bulletin, abstract 47-7-296, also abstract 47-9-358.

[2] See C. Jordan, *Calculus of Finite Differences*, p. 265.

[3] For usage of the name Euler-Maclaurin see footnote of a paper by the author this Bulletin, vol. 45 (1939), p. 748. Usage in the present instance is that of Nörlund, loc. cit., p. 29.

[4] See, for example, Nörlund, loc. cit., p. 17.