ON EQUIVALENCE OF CERTAIN TYPES OF SERIES OF ORTHONORMAL FUNCTIONS

G. H. PEEBLES

If one has a set of "properly" independent functions $u_1(x), u_2(x), \cdots, u_n(x), \cdots$ and a function $p(x)$ which on an interval (a, b) is non-negative, integrable and such that $\int_a^b p(x)dx > 0$, one can construct a second set $v_1(x), v_2(x), \cdots, v_n(x), \cdots$ $[v_n(x) = a_{n1}u_1(x) + \cdots + a_{nn}u_n(x)$, where a_{nk} is a constant$]$ whose members satisfy the relations

$$\int_a^b p(x)v_n(x)v_m(x)dx = \delta_{nm}.$$

Associated with such sets of orthonormal functions is the problem of expanding an "arbitrary" function $f(x)$ in a convergent series of the members of a particular set, that is, the problem of determining conditions sufficient for

$$\lim_{n\to\infty} \sum_{k=1}^{n} v_k(x) \int_a^b p(t)f(t)v_k(t)dt = f(x). \tag{1}$$

One approach to the problem lies in showing that expansion (1) is equivalent to another expansion, usually a Fourier series, whose behavior is known.[1] In the case of orthogonal polynomials, where $u_n(x) = x^{n-1}$, the writer was able to show, under conditions not too restrictive, that the expansions of a function in terms of the two sets of orthogonal polynomials corresponding, respectively, to different weight functions converge to the same value or diverge together.[2] It is the purpose of this note to point out that similar results can be obtained for systems of orthonormal functions constructed from a set $[u_n(x)]$ which has the property that the product of any two members

Presented to the Society, September 7, 1939; received by the editors August 7, 1941. Results presented to the Society, April 14, 1939 under the title *The boundedness of certain systems of orthogonal functions*, are implicit in the brief discussion on boundedness of orthonormal functions which appears at the end of this paper.

[1] Theorems and references on equiconvergence for orthogonal polynomials are given by G. Szegö, *Orthogonal Polynomials*, American Mathematical Society Colloquium Publications, vol. 23, 1939, chaps. 9 and 13. For equivalence for another type of orthogonal functions, see J. Mercer, *Sturm-Liouville series of normal functions in the theory of integral equations*, Philosophical Transactions of the Royal Society of London, (A), vol. 211 (1912), pp. 111–198; pp. 174–177.

[2] *An equivalence theorem for series of orthogonal polynomials*, Proceedings of the National Academy of Sciences, vol. 3 (1939), pp. 97–104.