NOTE ON CLOSURE FOR ORTHOGONAL POLYNOMIALS

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In this short note we give a simple theorem concerning closure for orthogonal polynomials (OP). The importance of closure in the theory of OP, particularly in the study of expansions of functions in series of OP, can hardly be overestimated.

The notations employed below are those of my monograph, *Théorie générale des polynômes orthogonaux de Tchebicheff* (Mémorial des Sciences Mathématiques, vol. 66, 1934), referred to as M.

We write generally

$$f(x) \sim \sum_{n=0}^{\infty} f_n \phi_n(x)$$

to signify that

$$f_n = \int_a^b f(x)\phi_n(x)d\psi, \qquad n = 0, 1, 2, \cdots,$$

$$\phi_n(x) \equiv \phi_n(x; a, b; d\psi),$$

$$\int_a^b \phi_m(x)\phi_n(x)d\psi = \delta_{mn}, \quad m, n = 0, 1, 2, \cdots.$$

The hereafter assumed existence of $\int_a^b f^2(x)d\psi$ implies, as is known, the convergence of the series $\sum_{n=0}^{\infty} f_n^2$. By closure we mean the following equality, Parseval's formula:

$$\int_a^b f^2(x)d\psi = \sum_{n=0}^{\infty} f_n^2.$$

THEOREM. *If closure holds for the "symmetric" sequence*

$$(1) \qquad \{\phi_n(x; -h, h; d\psi)\}, \qquad \psi(\infty) - \psi(x) \equiv \psi(-x)$$

[or

$$\{\phi_n(x; -h, h; p(x))\}, \qquad p(x) \equiv p(-x)]$$

then each of the following two sequences of OP is closed

$$(2) \qquad \{\phi_n(x; 0, h^2; d\psi_1)\}, \qquad \{\phi_n(x; 0, h^2; d\psi_2)\}$$
$$d\psi_1(x) = 2d\psi(x^{1/2}), \qquad d\psi_2(x) = xd\psi_1(x)$$

[or

Received by the editors October 2, 1941.